A fundamental question

Strong Virus

Epidemic?
example (static graph)

Weak Virus

Epidemic?
Problem Statement

Find, a condition under which

– virus will die out exponentially quickly
– regardless of initial infection condition
Threshold (static version)

Problem Statement

- **Given:**
  - Graph G, and
  - Virus specs (attack prob. etc.)

- **Find:**
  - A condition for virus extinction/invasion
Threshold: Why important?

- Accelerating simulations
- Forecasting (‘What-if’ scenarios)
- Design of contagion and/or topology
- A great handle to manipulate the spreading
  - Immunization
  - Maximize collaboration

…..
Q: What is the epidemic threshold?

- Background
- Result and Intuition (Static Graphs)
- Proof Ideas (Static Graphs)
- Bonus: Dynamic Graphs
“SIR” model: life immunity (mumps)

- Each node in the graph is in one of three states:
  - Susceptible (i.e. healthy)
  - Infected
  - Removed (i.e. can’t get infected again)
Related Work


All are about either:

- **Structured topologies** (cliques, block-diagonals, hierarchies, random)
- **Specific virus propagation models**
- **Static graphs**

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Q: What is the epidemic threshold?

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How should the answer look like?

- Answer should depend on:
  - Graph
  - Virus Propagation Model (VPM)

- But how??
  - Graph – average degree? max. degree? diameter?
  - VPM – which parameters?
  - How to combine – linear? quadratic? exponential?

\[ \beta d_{avg} + \delta \sqrt{\text{diameter}} \quad ? \quad (\beta^2 d_{avg}^2 - \delta d_{avg}) / d_{max} \quad ? \quad \ldots \]
Static Graphs: Main Result [Prakash+, 2011]

- Informally,

For,

- any arbitrary topology (adjacency matrix A)
- any virus propagation model (VPM) in standard literature

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( C_{VPM} )</th>
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<tr>
<td>No epidemic if ( \lambda \cdot C_{VPM} &lt; 1 )</td>
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the epidemic threshold depends only
1. on the \( \lambda \), first eigenvalue of \( A \), and
2. some constant \( C_{VPM} \), determined by the virus propagation model
Our thresholds for some models

- \( s = \text{effective strength} \)
- \( s < 1 : \text{below threshold} \)

<table>
<thead>
<tr>
<th>Models</th>
<th>Effective Strength ((s))</th>
<th>Threshold (tipping point)</th>
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<tbody>
<tr>
<td>SIS, SIR, SIRS, SEIR</td>
<td>( s = \lambda \cdot \left( \frac{\beta}{\delta} \right) )</td>
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<tr>
<td>SIV, SEIV</td>
<td>( s = \lambda \cdot \left( \frac{\beta \gamma}{\delta (\gamma + \theta)} \right) )</td>
<td>( s = 1 )</td>
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<tr>
<td>SI(_1)I(_2)V(_1)V(_2) (H.I.V.)</td>
<td>( s = \lambda \cdot \left( \frac{\beta_1 v_2 + \beta_2 \epsilon}{v_2 (\epsilon + v_1)} \right) )</td>
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Our result: Intuition for $\lambda$

**“Official” definition:**
- Let $A$ be the adjacency matrix. Then $\lambda$ is the root with the largest magnitude of the characteristic polynomial of $A \ [\text{det}(A - xI)]$.

- Doesn’t give much intuition!

**“Un-official” Intuition 😊**
- $\lambda \sim \# \text{ paths in the graph}$

\[
A_k^{(i,j)} = \# \text{ of paths } i \rightarrow j \text{ of length } k
\]

\[
A_k = \begin{array}{ccc}
\vdots \\
0 \\
\vdots \\
A_0 \\
\end{array}
\]

$u \quad u$
Largest Eigenvalue ($\lambda$)

better connectivity $\rightarrow$ higher $\lambda$

$\lambda \approx 2$

(a) Chain

$\lambda = \sqrt{N}$

(b) Star

$\lambda = N - 1$

(c) Clique

$N = 1000$

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Examples: Simulations – SIR (mumps)

(a) Infection profile

(b) “Take-off” plot

PORTLAND graph

31 million links, 6 million nodes
Examples: Simulations – SIRS (pertussis)

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Q: What is the epidemic threshold?

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Proof Sketch

Model-based

Graph-based

\[ \lambda * C_{VPM} < 1 \]

General VPM structure

Topology and stability

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(A) Unstable
(B) Stable
(C) Neutral (at threshold)

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Some trivia

- first person in the US identified as a healthy carrier of the pathogen associated with typhoid fever.
- infected some 53 people, over the course of her career as a cook!
- forcibly quarantined by public health authorities
Two “Infected” States

Asymptomatic

Symptomatic

I_1

I_2

Sneezing

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Ingredient 1: Our generalized model

\[ S^* I^2 V^* \quad (S^* I^* V^*) \]

Diagram:
- Susceptible (S\(_1\), S\(_2\), ...)
- Endogenous Transitions
- Infected (I\(_1\), I\(_2\))
- Exogenous Transitions
- Vigilant (V\(_1\), V\(_2\), ...)
- Endogenous Transitions

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## Models and more models

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\( S_1 I_2 V_1 V_2 \)  H.I.V.
Our generalized model

\[ S^*I^2V^* \ (S^*I^*V^*?) \]
Special case: SIR
Special case: H.I.V.

SI₁₁₂₂VV₁₂

“Non-terminal”

“Terminal”

Multiple Infectious, Vigilant states
Ingredient 2: NLDS + Stability

- View as a NLDS
  \[ \vec{P}_{t+1} = \mathcal{G}(\vec{P}_t) \]
  - discrete time
  - non-linear dynamical system (NLDS)

**Probability vector**

- Specifies the **state of the system** at time \( t \)
- \( size N \) (number of nodes in the graph)
Ingredient 2: NLDS + Stability

- View as a NLDS
  \[ \vec{P}_{t+1} = G(\vec{P}_t) \]
  - discrete time
  - non-linear dynamical system (NLDS)

Non-linear function
Explicitly gives the evolution of system

\[ G : \mathbb{R}^{mN} \rightarrow \mathbb{R}^{mN} \]
Ingredient 2: NLDS + Stability

- View as a NLDS
  - discrete time
  - non-linear dynamical system (NLDS)

- Threshold $\rightarrow$ Stability of NLDS
Special case: SIR

\[ \vec{P}_{t+1} = G \vec{P}_t \]

where

\[ G : \mathbb{R}^{3N} \rightarrow \mathbb{R}^{3N} \]

is a function that defines the transition probabilities between states S, I, and R.

- \( P_{S,i,t+1} = P_{S,i,t} \zeta_i(t)(I) \)
- \( P_{I,i,t+1} = P_{S,i,t}(1 - \zeta_i(t)(I)) + (1 - \delta)P_{I,i,t} \)
- \( P_{R,i,t+1} = \delta P_{I,i,t} + P_{R,i,t} \)

\[ \zeta_i(t)(I) = \text{probability that node } i \text{ is not attacked by any of its infectious neighbors} \]
Fixed Point

State when no node is infected

Q: Is it stable?
Stability for SIR

Stable
under threshold

Unstable
above threshold
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**General VPM structure**

See paper for full proof

**Model-based**

$\lambda^* C_{VPM} < 1$

**Graph-based**

Topology and stability

(A) Unstable

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Outline

- Q: What is the epidemic threshold?
  - Background
  - Result and Intuition (Static Graphs)
  - Proof Ideas (Static Graphs)
  - Bonus: Dynamic Graphs
Dynamic Graphs: Epidemic?

DAY (e.g., work)

Alternating behaviors
Dynamic Graphs: Epidemic?

NIGHT (e.g., home)

Alternating behaviors

adjacency matrix
Model Description

- **SIS model**
  - recovery rate $\delta$
  - infection rate $\beta$

- **Set of $T$ arbitrary graphs**

\[
\{ A_1, A_2, \ldots, A_T \}
\]

\[\text{day}\]

\[\text{night}\]

, weekend.....
Obvious result

- No epidemic if \( \frac{\lambda_{\text{max}} \beta}{\delta} < 1 \)

- BUT
  - Too pessimistic!
Main result: Dynamic Graphs Threshold [Prakash+, 2010]

- Informally, NO epidemic if

\[ \text{eig} (S) = \lambda_S < 1 \]

Single number! Largest eigenvalue of The system matrix \( S \)
NO epidemic if $eig(S) = \lambda_S < 1$

$$S = \prod_i S_i$$

$$S_i = (1 - \delta)I + \beta A_i$$

**Cure rate**

**Infection rate**

**Adjacency matrix**

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Infection-profile

$log(fraction\ infected)$

Synthetic

MIT Reality

Time
“Take-off” plots

Footprint (# infected @ “steady state”)

Synthetic

MIT Reality

\[ \lambda \prod_i S_i \text{ (log scale)} \]