CS 6604: Data Mining Large Networks and Time-series

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Lecture #5: Power Laws and Preferential Attachment
## Recap

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<th>Real Graphs</th>
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Definition

- \( P(x) = C \ x^{-a} \) \( (x \geq x_{\text{min}}) \)
- E.g. \( \text{prob( city population between } x \text{ and } dx) \)
For discrete variables

- $P(k) = C \ k^{-\alpha} \ (k > 0)$

- Or the Yule distribution

$$P(k) = CB(k, a)$$

$$B(k, a) = \frac{\Gamma(k)\Gamma(a)}{\Gamma(k + a)} \approx k^{-a}$$
Power laws, Pareto distributions and Zipf's law


 FIG. 2 Left: histogram of the populations of all US cities with 10000 or more. Right: another histogram of the same data, but plotted on logarithmic scales. The approximate straight-line form of the histogram in the right panel implies that the distribution follows a power law. Data from the 2000 US Census.

Power-law distributions occur in an extraordinarily diverse range of phenomena. In addition to city populations, the sizes of earthquakes [3], moon craters [4], solar flares [5], computer files [6] and wars [7], the frequency of use of words in any human language [2, 8], the frequency of occurrence of personal names in most cultures [9], the numbers of papers scientists write [10], the number of citations received by papers [11], the number of hits on web pages [12], the sales of books, music recordings and almost every other branded commodity [13, 14], the numbers of species in biological taxa [15], people's annual incomes [16] and a host of other variables all follow power-law distributions.

Power laws also occur in many situations other than the statistical distributions of quantities. For instance, Newton's famous $1/r^2$ law for gravity has a power-law form with exponent $\alpha = 2$. While such laws are certainly interesting in their own way, they are not the topic of this paper. Thus, for instance, there has in recent years been some discussion of the "allometric" scaling laws seen in the physiognomy and physiology of biological organisms [17], but since these are not statistical distributions they will not be discussed here.

Readers interested in pursuing the subject further may also wish to consult the reviews by Sornette [18] and Mitzenmacher [19], as well as the bibliography by Li.
Power laws, Pareto distributions and Zipf's law

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Exponential vs Power-Laws

- Exponential decays much faster!
- Power Laws have longer ‘tails’

Mathematical expressions:

- Exponential: $p(x) = cx^{-0.5}$
- Power Law: $p(x) = cx^{-1}$
Degree Distribution of Web is heavily Skewed

\[ P[k] \approx k^{-\alpha}, \ 2 < \alpha < 3 \]

Broder et al 2000
AS routers [Faloutsos 1999]

\[ P[k] \approx k^{-\alpha}, \ 2 < \alpha < 3 \]
Other Networks [Barabasi, Albert 1999]

\[ P[k] \approx k^{-\alpha}, \quad 2 < \alpha < 3 \]
Power-laws are everywhere

[Clauset et al. 2007]
Exploiting Long Tails

ANATOMY OF THE LONG TAIL

Online services carry far more inventory than traditional retailers. Rhapsody, for example, offers 19 times as many songs as Wal-Mart's stock of 39,000 tunes. The appetite for Rhapsody's more obscure tunes (charted below in yellow) makes up the so-called Long Tail. Meanwhile, even as consumers flock to mainstream books, music, and films (right), there is real demand for niche fare found only online.

Sources: Erik Brynjolfsson and Jeffrey Hu, MIT, and Michael Smith, Carnegie Mellon; Barnes & Noble; Netflix; RealNetworks

C. Andersen, WIRED, 2004
Some examples which are NOT P.Ls

\[ x^{\alpha} \cdot e^{-bx} \]

\[ x^{\alpha} \cdot \text{constant} \]

\[ x^{\alpha} \]

\[ x^{\alpha} \cdot \text{constant} \cdot \text{log}(x) \]

\[ x^{\alpha} \cdot \text{constant} \cdot \text{log}(x)^2 \]
Binomial vs Power-Law Degree Distribution

Bell Curve
- Most nodes have the same number of links
- No highly connected nodes

Power Law Distribution
- Very many nodes with only a few links
- A few hubs with large number of links

Number of nodes with k links vs Number of links (k)

Prakash 2017
CS 6604: DM Large Networks & Time-Series
Scale Free Networks

- Networks with power-law tails in degree distribution

- Name comes from Physics
  - Scale invariance: no characteristic scale
  - $f(a \cdot x) = a^c f(x) \implies A$ scale free function
    - Verify it holds in power-laws
Heavy-Tailed Distributions

- Formally, $P(x)$ is heavy tailed if
  
  $$
  \lim_{x \to \infty} \frac{P(X > x)}{e^{-\lambda x}} = \infty
  $$

- Exponential:
  
  $$
  P(X) = \lambda e^{-\lambda x}
  $$
  
  $$
  P(X > x) = 1 - P(X \leq x) = e^{-\lambda x}
  $$

- Normal:
  
  $$
  P(X) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{(x-\mu)^2}{2\sigma^2}}
  $$

Both are not heavy-tailed
Heavy Tail Examples: AKA Long Tails, Zipf’s Law, Pareto’s Law etc.

- Power Law: $P(x) \propto x^{-\alpha}$
- Power Law with Cutoff: $x^{-\alpha} e^{-\lambda x}$
- Stretched Exponential: $x^{\beta-1} e^{-\lambda x^\beta}$
- Log-Normal: $\frac{1}{x} \exp \left[ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right]$
Normalizing Constant for Power-Laws

- **Q:** $p(x) = cx^{-\alpha}$, with $x > x_{min}$ what is $c$?

- **See that:**

\[
\int_{x \geq x_{min}} p(x) \, dx = 1
\]

\[
c \int_{x \geq x_{min}} x^{-\alpha} = 1
\]

\[
\Rightarrow c = (\alpha - 1)x_{min}^{\alpha-1}
\]

\[
p(x) = \frac{\alpha - 1}{x_m} \left(\frac{x}{x_m}\right)^{-\alpha}
\]
Average value? $E[X]$ of a power-law

- $P(x)$ is a power-law

\[
E[x] = \int_{x_m}^{\infty} x \, p(x) \, dx = z \int_{x_m}^{\infty} x^{-\alpha+1} \, dx
\]

\[
= - \frac{(\alpha-1)x_m^{\alpha-1}}{2-\alpha} \left[ \infty^{2-\alpha} - x_m^{2-\alpha} \right]
\]

Need: $\alpha > 2$

\[
E[X] = \frac{\alpha - 1}{\alpha - 2} x_m
\]
Infinite Moments

- **Average**: \( E[X] = \frac{\alpha - 1}{\alpha - 2} x_m \)  
  (if \( \alpha \leq 2 \), \( E[X] \) is \( \infty \))

- **Variance**: if \( \alpha \leq 3 \), \( \text{Var}[X] = \infty \)

In real networks \( 2 < \alpha < 3 \) so:  
\( E[x] = \text{const} \)  
\( \text{Var}[x] = \infty \)
Three Versions of P.L.

PDF
= frequency-count plot

Zipf plot
= rank-frequency plot

NCDF = CCDF
(Complementary CDF)

USEFUL!

IF ONE PLOT IS A P.L., SO ARE THE OTHER TWO

Prob (area = x)

-\(a-1\)

Prob (area >= x)

-\(a\)

area

-\(1/a\)

rank

x
Estimating $\alpha$ from data

- One way: Fit a line on the log-log plot using least squares i.e. solve $\arg\min_{\alpha} (\log(y) - \alpha \log(x))^2$
Better way: Use the three versions

- Plot CCDF: $P(X \geq x)$. Then estimated $\alpha = 1 + \alpha'$
  - Try to prove it!
Case-Study: FlickR

Log scale, $\alpha=1.75$

CCDF, Log scale, $\alpha=1.75$

CCDF, Log scale, $\alpha=1.75$, exp. cutoff
Another way: Max. Likelihood Estimation

\[ p(x) = \frac{\alpha - 1}{x_m} \left( \frac{x}{x_m} \right)^{-\alpha} \]

\[ L(\alpha) = \ln(\prod_i^n p(d_i)) = \sum_i^n \ln p(d_i) \]

\[ = \sum_i^n \ln(\alpha - 1) - \ln(x_m) - \alpha \ln \left( \frac{d_i}{x_m} \right) \]

- So we want \( \alpha^* = \arg \max_{\alpha} L(\alpha) \)

\[ \frac{dL(\alpha)}{d\alpha} = 0 \Rightarrow \frac{n}{\alpha - 1} - \sum_i \ln \left( \frac{d_i}{x_m} \right) = 0 \]

\[ \alpha^* = 1 + n \left[ \sum_i^n \ln \left( \frac{d_i}{x_m} \right) \right]^{-1} \]
Max. Expected Degree in a Scale-Free network

- Let $K$ be the max. expected degree $\rightarrow$ the expected number of nodes with degree $> K$ should be $< 1$

$$
\int_{K}^{\infty} p(x) \, dx \approx \frac{1}{n}
$$

$$
= \frac{(\alpha-1)x_m^{\alpha-1}}{-(\alpha-1)} \left[ 0 - K^{1-\alpha} \right] = x_m^{\alpha-1} \cdot K^{-(\alpha-1)} \approx \frac{1}{n}
$$

$$
K = x_m n^{1/(\alpha-1)}
$$
Max. Expected Degree: Consequences

- Q: Why don’t we see networks in real-life with $\alpha = 4, 5, 6...$?
- In real networks $K \approx 1000$
- How large should $n$ be if $\alpha = 4, 5, 6...$?
  - If $\alpha = 5$, 
    $$K = x_m \frac{1}{n^{\alpha-1}}$$
    $$n = \left(\frac{K}{x_m}\right)^{\alpha-1} \approx 10^{12}$$
TIME-OUT: SOME INFO
### Course Grading: Updated

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<tr>
<th>Component</th>
<th>Weight</th>
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<tr>
<td>Homework</td>
<td>10%</td>
<td>One</td>
</tr>
<tr>
<td>Presentation</td>
<td>15%</td>
<td>Presenting a paper in class</td>
</tr>
<tr>
<td>Course project</td>
<td>60%</td>
<td>BIGGEST component</td>
</tr>
<tr>
<td>Class Participation</td>
<td>10%</td>
<td>!= attendance 😊</td>
</tr>
<tr>
<td>‘Scribe’</td>
<td>5%</td>
<td>3 paragraph reaction posts on Piazza</td>
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#### Class participation necessary!
- **To get at least 5%**
  - Non-trivial response to ‘reaction’ posts [see next slide] at least 6 times
  - Remember Piazza gives post statistics 😊
Scribe: Reaction Post

- A 3 paragraph post on piazza after each class
  - Due 10pm the day of the lecture
  - Summary (4 lines)
    - What was the main technical content?
    - How did it relate to other topics?
    - What were the main doubts raised by students?
  - Expanding (3 lines)
    - Describe least one non-trivial related aspect not covered in the lecture
  - Brainstorming (4 lines)
    - Any open interesting questions (e.g. which readings missed)? Extensions?
    - What possible applications could benefit from the proposed material?
Scribe: Reaction Post

- 1 scribe per lecture
  - Everyone would scribe one lecture
- Starts from Oct 16.
- Readings will be posted on Friday for the following week.
- The schedule will be posted on Piazza after lecture
  - Essentially will follow class-order alphabetically as on Hokiespa
Project (Groups of 2-3)

- Has to be **substantial**
- Can be
  - **Experimental**: evaluation of different algorithms and models on an interesting dataset(s)
  - **Theoretical**: considers a model (can be novel), or an algorithm, or a metric and derives a rigorous result about it (e.g. tighter bounds, surprising properties etc)
  - **Extension**: an extension or improvement of a method or model covered in class to a different or more general setting (eg. time-varying, distributed, anytime, scalable, etc.) and experiments that justify the new proposal
- Can **NOT** be just a survey
- (no double dipping etc. → already explained)
### Project

- **Deliverables**
  - Project Proposal: 15%
  - Project Milestone Report: 10%
  - Final Report: 20%
  - Final Presentation in class: 15%

- Proposal should contain a detailed survey of 3-4 pages
  - 6-8 papers in a topic of your interest
    - Discussed in or related to the course
Project

- Deliverables
  - Project Proposal
  - Project Milestone Report
  - Final Report
  - Final Presentation/Poster in class

- Start EARLY!

Rough due Dates
- Oct. 16
- Nov. 13
- Dec. 10
- Dec. 11/13
One Homework

- To be released on Sept. 20
- Due on Oct 2 (beginning of class)
- Theoretical + Hands-on

- Start EARLY!
Presentation/Grading

- Starting Oct 16 [roughly 1/2 time-point]
  - 1 person will present 1 lecture on a given topic (1-2 papers)
    - Create your own power-point slides
  - Will post Doodle link by last week of Sept for topic sign-up
  - Will mail exact papers/readings to the students 10 days in advance
- 12 lectures
Generative Processes for P.L.s

- CAN NOT arise from sums of independent event
- Central Limit Theorem

\[ X, X_1, ..., X_n : \text{rnd. vars with mean } \mu, \text{ variance } \sigma^2 \]
\[ S_n = \sum_i X_i : E[S_n] = n\mu, \text{ var}[S_n] = n\sigma^2, \text{ SD}[S_n] = \sigma \sqrt{n} \]

\[ P(S_n = E[S_n] + x \cdot \text{SD}[S_n]) \sim \frac{1}{2\pi} e^{-\frac{x^2}{2}} \]
Combination of exponentials

- Let $P(y) = e^{ay}$
  e.g. radioactive decay, with half life $-a$
  (= collection of people playing russian roulette)

- Let $X = e^{bY}$
  (every time a person survives, we double her earnings)

- Then $P(X) = P(Y) * \frac{dy}{dx}$
  \[ = \frac{1}{b} x^{(\frac{a}{b}-1)} \]
  [standard change of variables]
  [power-law]

i.e. the final capital of each person follows a P.L.
Combination of exponentials

- Monkey on a Typewriter
- \( N = 26 \) letters equiprobable
- Space bar has prob. \( q \)
- **THEN:** [Newman 2005]

Freq(x-th most frequent word) = \( x^{-a} \)

\[
a = \frac{2 \ln m - \ln(1 - q_s)}{\ln m - \ln(1 - q_s)}
\]
Random Walks

- What is the PDF of inter-arrival times?

\[ p(t) \sim ? \]
Random Walks

- What is the PDF of inter-arrival times?

\[ p(t) \sim t^{-1.5} \]

AKA Drunkard’s Walk 😊
Forest Fires and Percolation

- A burning tree will cause its neighbors to burn next.
- Which tree density $p$ will cause the fire to last longest?
Forest Fires and Percolation

Percolation and forest fires

A burning tree will cause its neighbors to burn next. Which tree density $p$ will cause the fire to last longest?
Forest Fires and Percolation

Percolation and forest fires

Density

Burning time

Percolation threshold, $p_c \approx 0.593$
Forest Fires and Percolation

Percolation and forest fires

Burning time

N

density

0

CMU SCS

Copyright: C. Faloutsos (2012)

Percolation and forest fires

At pc ~ 0.593:
'patches' of all sizes;
Korcak-like 'law'.
Forest Fires and Percolation

- At $pc \sim 0.593$
- No characteristic scale; ‘patches’ of all sizes;
Fragmentation

$p_1 \times \ldots \times (1-p_1)\ldots$

Resulting distributions of sizes: lognormal
Log-normals

Lognormal:

\[ \log(\text{pdf}) = \text{parabola} \]

\[ \log(\text{pdf}) \]

\[ \log(\text{\$}) \]
Log-normals

Lognormal:

log(pdf) vs. log($)

parabola

- Random multiplication
- Fragmentation
  -> lead to lognormals (~ look like power laws)

- Stick of length 1
- Break it at a random point x (0<x<1)
- Break each of the pieces at random
  -> Resulting distribution: lognormal (why?)
Yule Distribution and Chinese Restaurant Process

- Newcomer to a restaurant
  - Joins an existing table
  - Or starts a new table/group of its own, with prob. $\frac{1}{m}$
Then,

\[ P(\text{k people in a group } ) = p_k \]

\[ = (1 + 1/m) \, B(k, 2+1/m) \]

\[ \sim \, k^{-(2+1/m)} \]

\( (B(a, b) \sim a^{-b}: \text{a power-law tail}) \)
AKA Preferential Attachment for Graphs [Price’65, Albert-Barabasi’99]

- Nodes arrive in order 1, 2, 3, ... n
- At some step j, let $d_i$ be the degree of node $i < j$
- A new node $j$ arrives and creates $m$ out-links
- Prob. $(j \text{ connecting to node } i) \sim d_i$

$$P(j \rightarrow i) = \frac{d_i}{\sum_k d_k}$$
AKA Rich get Richer

- New nodes more likely to link to nodes that already have high degree
  - Herb. Simon’s result
    - PLs arise from cumulative advantage
  - Price’ 65
    - New citations to a paper are proportional to the number it already has
The exact model for analysis

[Mitzenmacher’03]

- Nodes arrive in order 1, 2,...n
- When node j is created it makes a single out-link to an earlier node i chosen:
  - With probability $p$, j links to i chosen uniformly randomly
  - With probability $1-p$, j does preferential attachment
    - i.e. j chooses a node i uniformly at random and links to a neighbor of node i
The model generates graphs where the fraction of nodes with in-degree $k$ scales as:

$$P(d_i = k) \propto k^{-\frac{2-p}{1-p}}$$

Skipping the proof

– Based on analyzing the ‘continuous degree’ $d_i(t)$ of a node
Preferential Attachment

- Intuitively reasonable process
- Can tune $p$ to get the observed exponent
  - For the web
    - Prob. $[k] \sim d^{-2.1}$
    - $2.1 = (2-p)/(1-p) \Rightarrow p \sim 0.1$
- BUT, there are other network formation mechanisms that generate scale-free networks
  - Copying model (Kleinberg+)
  - Forest Fire (Leskovec+)
  - Random Surfer (Blum+)
  - ......
  - [SIMILAR TO/BASED ON the ones we saw for P.L.s in general]
Preferential Attachment

- Two changes from $G(n,p)$
  - Growth
  - Preferential Attachment

- Do we need both?
  - YES
  - Let’s add growth to $G(n,p)$
    - $x_j = \text{degree of node } j \text{ at the end}$
    - $X_j(u) = 1$ if $u$ links to $j$, else 0
    - $X_j = X_j(j+1) + X_j(j+2) \ldots + X_j(n)$
    - $E[X_j(u)] = P[u \rightarrow j] = 1/(u-1)$
    - $E[X_j] = H_{n-1} - H_j \sim \log(n-1) - \log(j)$ NOT $(n/j)^a$  

[Harmonic Numbers]
PA: problems

- PA not so good at predicting network structure
  - Age-degree correlation [Adamic, Huberman]
  - Links among high-degree nodes
    - On the web nodes sometime avoid linking to each other
- Further questions
  - What is a reasonable model for how people sample through web-pages and link to them?
    - Short random walks
    - Effect of search engines: reach pages based on # of links
Size of the biggest hub is of order $O(N)$. Most nodes can be connected within two steps, thus the average path length will be independent of the network size.

The average path length increases slower than logarithmically. In $G_{np}$ all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network vast majority of the path go through the few high degree hubs, reducing the distances between nodes.

Some models produce $\alpha = 3$. This was first derived by Bollobas et al. for the network diameter in the context of a dynamical model, but it holds for the average path length as well.

The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.
Summary: Scale-Free networks

- **$\alpha = 1$**: Second moment $\langle k^2 \rangle$ diverges
- **$\alpha = 2$**: Average $\langle k \rangle$ diverges
- **$\alpha = 3$**: Ultra small world behavior

- Regime full of anomalies
  - The scale-free behavior is relevant
  - Behaves like a random network

Source: J. Leskovec
Application

"The anomalous bump on the x-axis is due to a large clique formed by a single spammer"
FRACTALS AND POWER-LAWS

Based on some material from Prof. C. Faloutsos
What is a fractal?

= self-similar point set, e.g. Sierpinski triangle

zero area;
infinite length!
Dimensionality?

- Paradox: Infinite perimeter; Zero area!

- ‘dimensionality’: between 1 and 2
  - Actually log(3)/log(2) = 1.58
Fractals

- Seminal work by Hilbert, Minkowski, Mandelbrot, Cantor, Ken Wilson, Lyapunov....
Fractals in nature
Fractals in nature 😊
Role of scaling [Mandelbrot]

11.5 \times 200 = 2300 \text{km}

28 \times 200 = 2800 \text{km}

70 \times 50 = 3500 \text{km}
Role of scaling

Usual scaling and dimension

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<th>N</th>
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<td>4</td>
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<td>3</td>
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<td>9</td>
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Koch Snowflake

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<tr>
<td>4</td>
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<tr>
<td>27</td>
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</table>
Fractal Dimension

- \( N \) = number of new sticks
- \( \varepsilon \) = scaling factor

\[ N \propto \varepsilon^{-D} \]

- \( D \) = fractal dimension

- Power-Law!
Another way to see it

- Correlation Fractal Dimension
  
  - Q: fractal dimension of a line?
  - A: \( \text{nn} ( \leq r ) \sim r^{1} \)
  
  (‘power law’: \( y=x^a \))

  
  - Q: fd of a plane?
  - A: \( \text{nn} ( \leq r ) \sim r^{2} \)
  
  \( \text{fd} = \text{slope of (log(nn) vs log (r))} \)
**Correlation Fractal Dimension**

- Avg nn (<= r) = total #pairs (<= r)/N

![Graph showing log(#pairs within <=r) vs. log(r)]

- log(#pairs within <=r) vs. log(r)
- 1.58

== ‘correlation integral’

---

**Observations:**
- Euclidean objects have integer fractal dimensions:
  - point: 0
  - lines and smooth curves: 1
  - smooth surfaces: 2
- Fractal dimension -> roughness of the periphery
Networks?

Internet routers: how many neighbors within h hops?

Reachability function: # of neighbors within r hops VS r

\[ \log(\text{pairs}) \]

\[ \log(\text{hops}) \]

2.8

F^3, SIGCOMM 1999
So are real networks Fractals?

- Answer: probably
- Multiple generators which mimic ‘fractal’ growth
  - R-MAT [Chakrabarti, Faloutsos]
  - Kronecker [Leskovec+]

  - Very good match to observed statistical properties
Fractals can be very useful

- Galaxies
- Brain scans

- Traffic

Fractals & power laws: appear in numerous settings:
- medical
- geographical / geological
- social
- computer-system related