Efficient Near-Optimal Control of Large-Size Second-Order Linear Time-Varying Systems

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Outline

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Introduction

- Optimal control of linear time-varying (LTV) systems is an active area of research
  - Rockets, robotics, structures, and others.
- Generally, these works deal with first-order LTV ordinary differential equations (ODEs).
- However, many engineering systems are described by second-order LTV ODEs:

$$\mathcal{M}(t)\ddot{q}(t) + \mathcal{C}(t)\dot{q}(t) + \mathcal{K}(t)q(t) = \mathcal{B}(t)u(t), \qquad (1)$$

  - space structures, spring-mass-damper systems, robotic manipulators, etc.
- This work focuses on optimal control of LTV second-order systems over finite horizon.

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Two-boundary Optimal Control

- Consider the second-order LTV system:

$$\mathcal{M}(t)\ddot{q} + \mathcal{C}(t)\dot{q} + \mathcal{K}(t)q = \mathcal{B}(t)u \tag{2}$$

- With the first-order form:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & I \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}\mathcal{C} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{0}_{n\times n} \\ \mathcal{M}^{-1}\mathcal{B} \end{bmatrix}}_{\bar{B}} u \tag{3}$$

- Control objective: design $u(t)$ to drive $x(t)$ from $x(0) = x_0 = [q_0 \ \dot{q}_0]^\top \rightarrow \mathrm{x(T)} = \mathrm{x}_T = [q_T \ \dot{q}_T]^\top$ within a prescribed T,

- Simultaneously minimizing the cost function:

$$J = \tfrac{1}{2}\int_0^T x^\top(t)Q(t)x(t) + u^\top(t)R(t)u(t) \ \ dt, \tag{4}$$

Two-boundary Optimal Control

- Generally addressed using the time-varying Hamiltonian-Jacobi-Bellman equation:

$$\mathcal{H} = x^\top(t)Q(t)x(t) + u^\top(t)R(t)u(t) + p^\top(t)(A(t)x(t) + B(t)u(t)), \tag{5}$$

- $p(t)$ is the co-state variable that satisfies

$$\dot{p}(t) = -\nabla_x \mathcal{H} = -Q(t)x(t) - A^\top(t)p(t) \tag{6}$$

- Optimal control law $u(t)$ ($\nabla_u \mathcal{H} = 0$):

$$u(t) = -R^{-1}(t)B^\top(t)p(t) \tag{7}$$

- Difficult to solve $\rightarrow$ numerically solved with Pontyagrin Maximum Principle (PMP)
  - PMP is computationally demanding!
- Interestingly, two-value boundary problems exhibit a two-time scale phenomenon.

Two-boundary Optimal Control

- Time scale decomposition can be used to obtain singularly perturbed system form[1].
- Possible to approximate the original LTV system with two LTI systems[1]:
  - Initial regulator problem (IRP) → solved in forward time
  - Terminal regulator problem (TRP) → solved in backward time

Figure 1: Two-value boundary problem

[1] Petar Kokotović, Hassan K Khalil, and John O'reilly. *Singular perturbation methods in control: analysis and design*. SIAM, 1999.

Two-boundary Optimal Control

- Let

$$\tau = \frac{t}{T}, \quad \varepsilon = \frac{1}{T}, \quad \alpha = \frac{\tau}{\varepsilon}, \quad \beta = \frac{1-\tau}{\varepsilon} \tag{8}$$

Initial Regulator Problem

$$\frac{d}{d\alpha} x_a(\alpha) = A(0)x_a(\alpha) + B(0)u_a(\alpha), \quad x_a(0) = x_0, \tag{9}$$

with the feedback controller u_a in the form

$$u_a(\alpha) \triangleq -K_a x_a(\alpha) = -R(0)B^\top(0)P_a(0)x_a(\alpha), \tag{10}$$

where $P_a(0)$ is the positive semidefinite solution of

$$A^\top(0)P_a(0) + P_a(0)A(0) - P_a(0)B(0)R(0)B^\top(0)P_a(0) + Q(0) = 0, \tag{11}$$

which minimizes the cost function

$$J(x_a, u_a) = \int_0^\infty x_a^\top Q(0)x_a + u_a^\top R(0)u_a \, d\alpha \tag{12}$$

Two-boundary Optimal Control

Terminal Regulator Problem

$$\frac{d}{d\beta}x_b(\beta) = -A(1)x_b(\beta) - B(1)u_b(\beta), \; x_b(0) = x_T, \tag{13}$$

can be obtained with the feedback controller

$$u_b(\beta) \triangleq -K_b x_b(\beta) = R(1)B^\top(1)P_b(1)x_b(\beta), \tag{14}$$

where $P_b(1)(= -N(1))$ is the positive semidefinite solution of

$$-A^\top(1)P_b(1) - P_b(1)A(1) - P_b(1)B(1)R(1)B^\top(1)P_b(1) + Q(1) = 0, \tag{15}$$

$$A^\top(1)N(1) + N(1)A(1) - N(1)B(1)R(1)B^\top(1)N(1) + Q(1) = 0 \tag{16}$$

which minimizes the cost function

$$J(x_b, u_b) = \int_0^\infty x_b^\top Q(1)x_b + u_b^\top R(1)u_b \, d\beta \tag{17}$$

Two-boundary Optimal Control

- Then, the approximate near-optimal solution is given by[2]:

$$x(\tau) = x_a(\alpha) + x_b(\beta) + \mathcal{O}(\varepsilon) \tag{18}$$

- The solution requires solving two CAREs ((11) and (16)).
- Standard methods inaccurate and computationally inefficient as the system size increases.
- Conversion to first-order systems $\implies$ doubles system size $\implies$ exacerbates the problem!
- These are not closed-form $\implies$ solved numerically; not parameterized in terms of $\mathcal{M}$, $\mathcal{C}$, $\mathcal{K}$.
- Hence, they do not provide an approximate closed-form solution to the optimal control problem of second-order LTV systems.

[2] Petar Kokotović, Hassan K Khalil, and John O'reilly. *Singular perturbation methods in control: analysis and design*. SIAM, 1999.

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Positive Semidefinite and Stabilizing Solution to IRP

Theorem

Let γ, a_1, and a_2 be any scalars such that

$$\gamma \in (0, \infty), \qquad a_1 > \frac{\gamma}{1+\gamma}\left(\lambda_{max}(\mathcal{M}(0)\mathcal{K}^{-3}(0))\right)^{\frac{1}{2}} > 0$$

$$a_2 \geq \lambda_{max}\left(\frac{\gamma}{2}\mathcal{C}^{\frac{-1}{2}}(0)(\mathcal{M}(0)\mathcal{K}^{-1}(0) + \mathcal{K}^{-1}(0)\mathcal{M}(0))\mathcal{C}^{\frac{-1}{2}}(0) + \frac{\gamma^2}{2(\gamma^2+2\gamma)}\mathcal{C}^{\frac{1}{2}}(0)\mathcal{K}^{-2}(0)\mathcal{C}^{\frac{1}{2}}(0)\right)$$

Then, a unique positive semidefinite solution to (11) is given below, where $a \geq max\{a_1,\ a_2\}$.

$$P(0) = \begin{bmatrix} (1+\gamma)a\mathcal{K}_0 & \gamma\mathcal{K}_0^{-1}\mathcal{M}_0 \\ \gamma\mathcal{M}_0\mathcal{K}_0^{-1} & a\mathcal{M}_0 \end{bmatrix} \tag{19}$$

$$Q(0) = \begin{bmatrix} (\gamma^2+2\gamma)I & \gamma\mathcal{K}_0^{-1}\mathcal{C}_0 \\ \gamma\mathcal{C}_0\mathcal{K}_0^{-1} & 2a\mathcal{C}_0 + a^2\mathcal{K}_0^2 - \gamma(\mathcal{M}_0\mathcal{K}_0^{-1} + \mathcal{K}_0^{-1}\mathcal{M}_0) \end{bmatrix} \tag{20}$$

$$R(0) = \mathcal{B}_0^\top\mathcal{K}_0^{-2}\mathcal{B}_0 \tag{21}$$

Negative Semidefinite and Destabilizing Solution

Theorem

Consider the system in (1), and let γ, a_1, and a_2 be any scalars such that

$$\gamma \in (0, \infty), \qquad a_1 > 2\lambda_{max}\left(\mathcal{K}^{-1}\mathcal{C}\mathcal{K}^{-1}\right),$$

$$a_2 \geq \lambda_{max}\left(\left[\gamma\mathcal{K}^{-1}(\mathcal{K}^{-1}\mathcal{M}+\mathcal{M}\mathcal{K}^{-1})\mathcal{K}^{-1}+\frac{\gamma}{\gamma+1}\mathcal{K}^{-2}\mathcal{C}^2\mathcal{K}^{-2}\right]^{\frac{1}{2}}\right)$$

Then, a unique negative semidefinite solution to (16) is given below, where $\bar{a} \geq max\{a_1,\ a_2\}$.

$$N(1) = \begin{bmatrix} -(1+\gamma)\bar{a}\mathcal{K}_1 & \gamma\mathcal{K}_1^{-1}\mathcal{M}_1 \\ \gamma\mathcal{M}_1\mathcal{K}_1^{-1} & -\bar{a}\mathcal{M}_1 \end{bmatrix} \tag{22}$$

$$Q(1) = \begin{bmatrix} (\gamma^2+2\gamma)I & \gamma\mathcal{K}_1^{-1}\mathcal{C}_1 \\ \gamma\mathcal{C}_1\mathcal{K}_1^{-1} & \bar{a}^2\mathcal{K}_1^2 - 2\bar{a}\mathcal{C}_1 - \gamma(\mathcal{M}_1\mathcal{K}_1^{-1}+\mathcal{K}_1^{-1}\mathcal{M}_1) \end{bmatrix} \tag{23}$$

$$R(1) = \mathcal{B}_1^\top\mathcal{K}_1^{-2}\mathcal{B}_1 \tag{24}$$

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Closed-form LTV solution

- Closed-loop matrix for IRP $\rightarrow$ closed-form stabilizing CARE solution:

$$\begin{aligned} A_{CL_i} &= A(0) - B(0)R^{-1}(0)B^{\top}(0)P_a(0) \\ &= \begin{bmatrix} \mathbf{0} & I \\ -(1+\gamma)\mathcal{M}_0^{-1}\mathcal{K}_0 & -\mathcal{M}_0^{-1}(\mathcal{C}_0 + a\mathcal{K}_0^2) \end{bmatrix} \end{aligned} \tag{25}$$

- LTI system $\implies$ $x_a(\alpha)$ directly obtained using the state-transition matrix.
- Similarly, the closed-loop matrix of the TRP $\rightarrow$ closed-form destabilizing CARE solution:

$$\begin{aligned} A_{CL_t} &= -\left(A(1) - B(1)R^{-1}(1)B^{\top}(1)N(1)\right) \\ &= \begin{bmatrix} \mathbf{0} & -I \\ (1+\gamma)\mathcal{M}_1^{-1}\mathcal{K}_1 & \mathcal{M}_1^{-1}(\mathcal{C}_1 - \bar{a}\mathcal{K}_1^2) \end{bmatrix} \end{aligned} \tag{26}$$

- Then, approximate closed-form solution $x(\tau)$:

$$x(\tau) = \exp\left(A_{CL_i}\frac{\tau}{\varepsilon}\right)x_0 + \exp\left(A_{CL_t}\frac{1-\tau}{\varepsilon}\right)x_f \tag{27}$$

Example: Spring-Mass-Damper Systems

- Drive system from $q_i(0) \rightarrow q_i(T)$ optimally.
- $\mathcal{M},\ \mathcal{C},\ \mathcal{K},\ \mathcal{B}$ LTV.
- Comparison between three methods:
  - PMP
  - Singular perturbation: Schur's method for CARE
  - Singular perturbation: Closed-form solution for CARE
- Comparison based on accuracy and computation time.

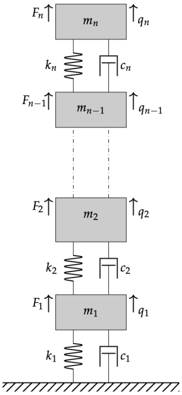

Figure 2: Spring-Mass-Damper System

Convergence

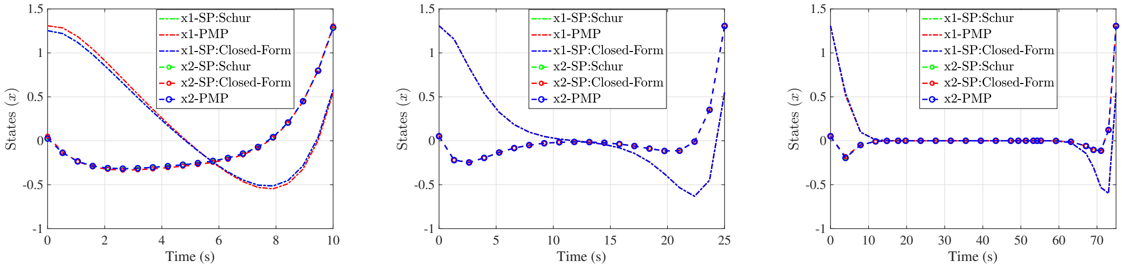

Figure 3: Trajectory for the states $x_1(t)$ and $x_2(t)$ for different values of $T = 10,\ 25,$ and 75 seconds.

- SP: Closed-Form solution converges towards the iterative solution (PMP) of the original LTV system as T increases ($\varepsilon \to 0$).

Convergence

Table 1: The root mean square error between the solutions of PMP and SP: Closed-form methods (SP:CF), as well as that between the PMP and the SP: Schur method (SP:S). The error is tabulated across the variation in system sizes as well as time (T).

System size	T = 10 sec		T = 25 sec		T = 75 sec	
	SP:CF	SP:S	SP:CF	SP:S	SP:CF	SP:S
1	0.3064	0.3079	0.1237	0.1243	0.0445	0.0448
5	0.6205	0.6230	0.2757	0.2771	0.0983	0.0989
10	1.1367	1.1425	0.5110	0.5136	0.1254	0.1248
50	1.2348	1.2348	0.5500	0.5500	0.1970	0.1970
100	1.3634	1.3632	0.6684	0.6679	0.2175	0.2168

Computation Time

- PMP solves the original LTV system numerically/iteratively → longest computation time.
- In contrast, the SP methods, SP: Closed-Form and SP: Schur, solve LTI systems → faster computation.
- SP: Closed-form faster than SP: Schur, as the latter solves CARE by determining Hamiltonian eigenvectors [2] → costly and inefficient with larger system sizes.
- Conversely, the SP: Closed-Form relies solely on elementary matrix operations!

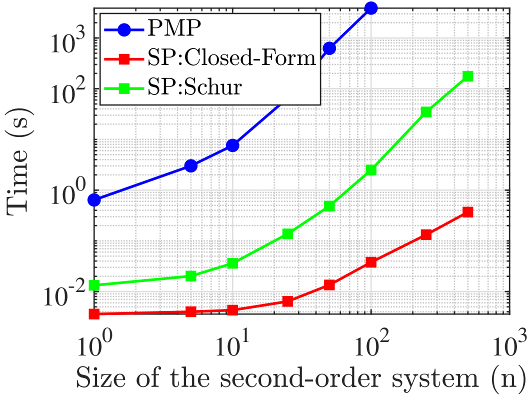

Figure 4: Logarithmic plot comparing the time taken to compute the solution by three methods: PMP, SP: Closed-Form, and SP: Schur for system sizes $n = 1,\ 5,\ 10,\ 25,\ 50,\ 100,\ 250,$ and 500.

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Conclusions

- Obtained an accurate and efficient approximate closed-form solution for the two-boundary optimal control problem of LTV second-order systems.
- Our approach involved decomposing the LTV problem into two LTI sub-problems.
- These sub-problems were solved using the proposed closed-form CARE solutions.
- Standard methods to solve these CAREs $\rightarrow$ inaccurate and computationally expensive solutions for large-size systems.
- In contrast, our closed-form solutions ensure accuracy and significantly reduce the computation cost for LTI second-order systems
- Consequently, the approximated LTV closed-loop system when compared with the standard numerical LTV solvers.

Thank You!!

Questions ??