A General Outer Bound for MISO Broadcast Channel with Heterogeneous CSIT

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Abstract—We study the impact of heterogeneity of channel-state-information available at the transmitters (CSIT) on the capacity of broadcast channels with a multiple-antenna transmitter and \( k \) single-antenna receivers (MISO BC). In particular, we consider the \( k \)-user MISO BC, where the CSIT with respect to each receiver can be either instantaneous/perfect (\( P \)), delayed (\( D \)), or not available (\( N \)); and we study the impact of this heterogeneity of CSIT on the degrees-of-freedom (DoF) of such network. We develop a general outer bound on the DoF region of \( k \)-user MISO BC for all possible heterogeneous CSIT configurations, assuming linear encoding strategies at the transmitter. The outer bound leads to an approximate linear sum-DoF characterization to within 0.5 for a broad range of CSIT configurations. It also leads to an exact characterization of linear sum-DoF for some specific CSIT configurations. Our proof of the outer bound relies on the development of a novel lemma called “Interference Decomposition Bound”, which lower bounds the interference dimension at a receiver which supplies delayed CSIT based on the average dimension of constituents of that interference, thereby decomposing it into its components.

I. INTRODUCTION

Channel state information at the transmitters (CSIT) plays a crucial role in the design and operation of multi-user wireless networks. The common procedure for obtaining channel state information (CSI) is to send training symbols (or pilots) at the transmitters, and then estimate the channels at the receivers and feed the estimates back to the transmitters. As a result of this feedback mechanism, CSI may not always be perfect and instantaneous. For instance, CSIT may be outdated due to slow feedback mechanism, it can be noisy (imperfect), or not available at all. Hence, one can expect various types ofCSI available at the transmitters with respect to different receivers in a large network. This results in communication scenarios with “heterogeneous” or “hybrid” CSIT.

As a result, there has been a growing interest in studying the impact of CSIT on the capacity of wireless networks, especially the broadcast channel. In particular, it was shown in [1] that even when the transmitter(s) only have access to delayed CSIT, there is significant potential for degrees-of-freedom (DoF) gain (also see subsequent works in [2]–[6]).

There have also been several works on studying the impact of heterogeneous (or hybrid) CSIT on the capacity of wireless networks, where the CSIT with respect to each receiver can now be either instantaneous/perfect (\( P \)), delayed (\( D \)), or not available (\( N \)) [7]–[11]. However, the only problem for which the DoF has been completely characterized for all hybrid CSIT configurations is the 2-user MISO BC, and the problem broadly remains open once we go beyond the 2-user scenario.

In this paper we consider the problem of \( k \)-user MISO BC with hybrid CSIT, where there is a multiple-antenna transmitter and \( k \) single-antenna receivers. The channels are time-varying, and the CSIT provided by each receiver is either instantaneous (\( P \)), delayed (\( D \)), or none (\( N \)). We provide a general outer bound on the DoF region when restricted to linear encoding schemes, which holds for all possible hybrid CSIT configurations. As a result, the bound provides an approximate linear sum-DoF characterization to within 0.5 for networks with more number of receivers supplying instantaneous CSIT than delayed CSIT. Furthermore, the bound leads to the exact characterization of linear sum-DoF for networks with only one receiver which supplies delayed CSIT.

The outer bound, which is the main contribution of this paper, is based on three main ingredients. The first is a novel lemma called “Interference Decomposition Bound”, which can be used to lower bound the interference dimension at a receiver with delayed CSIT by the average of dimensions of constituents of that interference, thereby decomposing the interference into its individual components.

The second main ingredient of the converse proof is the Least Alignment Lemma, presented in [12], [13], which states that once a receiver supplies no CSIT, the least amount of alignment will occur at that receiver, meaning that transmit signals will occupy the maximal signal dimensions at that receiver. Finally, the third ingredient gives a lower bound on the received signal dimension at a receiver supplying delayed CSIT based on received signal dimension at another receiver.

As for the achievability, we propose a multi-phase scheme exploiting side information from receivers supplying delayed CSIT while simultaneously harnessing instantaneous CSIT.

II. SYSTEM MODEL

We consider the Gaussian \( k \)-user multiple-input single-output broadcast channel (MISO BC) as depicted in Fig. 1. It consists of a transmitter with \( m \) antennas, and \( k \) single-antenna receivers, \( \text{Rx}_1, \text{Rx}_2, \ldots, \text{Rx}_k \), where \( m \geq k \). The transmitter has a separate message for each of the receivers.

\(^1\)The linearity constraint on encoding schemes is well motivated by the practical applicability and prevalence of linear precoding schemes.
We denote the set of indices of users in states the other two provide delayed CSIT.

where the first receiver provides instantaneous CSIT, while instantaneous CSIT with respect to Rx

denotes the channel coefficients of the channel from Tx to Rx$j$; and the additive noise is distributed as $z_j(t) \sim CN(0, 1)$. The elements of the channel coefficients vector $g_j(t)$ are i.i.d., drawn from a continuous distribution and also i.i.d across time and users. $G(t)$ denotes the set of all $k$ channel vectors at time $t$. In addition, we denote by $G^n$ the set of all channel coefficients from time 1 to $n$. Denoting the vector of transmit signals in a block of length $n$ by $\tilde{x}^n$, Tx obeys an average power constraint, $\frac{1}{n}E[\|\tilde{x}^n\|^2] \leq P_0$.

We focus on scenarios in which channel state information available at the transmitter (CSIT) with respect to different receivers can be instantaneous ($P$), delayed ($D$), or none ($N$). We refer to these scenarios as “hybrid scenarios”, or “hybrid” in short. In particular, CSIT with respect to Rx$j$, $j = 1, \ldots, k$, is denoted by $I_j \in \{P, N, D\}$, as defined in [9]. In this notation, $I_j = P$ indicates that Tx has access to instantaneous CSIT with respect to Rx$j$; i.e., at time $t$, Tx has access to $\{g_j(1), \ldots, g_j(t-1)\}$. Similarly, $I_j = D$ indicates delayed CSIT with respect to Rx$j$; i.e., at time $t$, Tx has access to $\{g_j(1), \ldots, g_j(t-1)\}$. Finally, $I_j = N$ indicates no CSIT. We assume that the type of CSIT for each receiver is fixed and does not alter over time (nevertheless, channels are time-varying). So, there are $3^k$ different fixed hybrid scenarios. As an example, we use PDD to denote the 3-user MISO BC where the first receiver provides instantaneous CSIT, while the other two provide delayed CSIT.

**Definition 1.** We denote the set of indices of users in states $P, D, N$ by $\mathcal{P}, \mathcal{D}, \mathcal{N}$, respectively. In addition, for a set $S$ we denote by $\pi_S$ a permutation of the elements of $S$, where the order of elements will become $\pi_S(1), \pi_S(2), \ldots, \pi_S(|S|)$.

Note that according to Definition 1, $P \cup D \cup N = \{1, 2, \ldots, k\}$ and $P \cap D = D \cap N = P \cap N = \emptyset$.

We restrict ourselves to linear coding strategies as defined in [4], [14], [15], in which DoF represents the dimension of the linear subspace of transmitted signals. More specifically, consider a communication scheme with block length $n$ in which Tx wishes to deliver a vector $\tilde{x}_j \in \mathbb{C}^{m_j(n)}$ of $m_j(n) \in \mathbb{N}$ information symbols to Rx$_j$ ($j \in \{1, 2, \ldots, k\}$). Each information symbol is a random variable with variance $P_0$. These information symbols are then modulated with precoding matrices $V_j(t) \in \mathbb{C}^{m \times m_j(n)}$ at times $t = 1, 2, \ldots, n$. Note that the precoding matrix $V_j(t)$ depends only upon the outcome of $\tilde{G}^j = \{G_j^i : i \in \mathcal{P} \cup \{G_j^{-1} \cap \mathcal{D}) \} \cup \{G_j^{-1} \cap \mathcal{D} \}$ due to the hybrid CSIT constraint:

$$V_j(t) = f_{j,t}^{(n)}(\tilde{G}^j),$$

(2)

Based on this linear precoding, Tx will then send $\tilde{x}(t) = \sum_{j=1}^{n} V_j(t) \tilde{x}_j$ at time $t$. We can rewrite $\tilde{x}(t)$ as $\tilde{x}(t) = [V_1(t) \ldots V_k(t)] [x_1; \ldots; x_k]$, where $[A; B]$ denotes the vertical concatenation of matrices $A$ and $B$ (i.e., $[A; B])$.

We denote by $V^n_j \in \mathbb{C}^{nm \times m_j(n)}$ the overall precoding matrix of Tx for Rx$_j$, such that the rows $1+(t-1)m, \ldots, tm$ of $V^n_j$ form $V_j(t)$. In addition, we denote the precoding function used by Tx by $f^{(n)} = \{f^{(n)}_{j,t}\}_{j=1,...,k}$.

Based on the above setting, the received signal at Rx$_j$ ($j \in \{1, 2, \ldots, k\}$) after the $n$ time steps of the communication will be $\tilde{y}_j^n = G_j^n [V_1^n \ldots V_k^n] [x_1; \ldots; x_k] + Z_j^n$, where $G_j^n \in \mathbb{C}^{m \times m_j(n)}$ is the block diagonal channel coefficients matrix where the channel coefficients of timeslots $t$ (i.e., $g_j^n(t)$) are in the row $t$, and in the columns $1+(t-1)m, \ldots, tm$ of $G_j^n$, and the rest of the elements of $G_j^n$ are zero.

Now, consider the decoding of $\tilde{x}_j$ at Rx$_j$ (i.e., the $m_j(n)$ information symbols for Rx$_j$). The corresponding interference subspace at Rx$_j$, $j \in \{1, 2, \ldots, k\}$, after the $n$ time steps of the communication will be $\tilde{W}_j^n = G_j^n [U] V_j^n$, where $[U]$ is the matrix formed by row concatenation of matrices $U_i^n$ for $i \neq j$, and $\text{colspan}([U])$ of a matrix corresponds to the sub-space that is spanned by its columns. Let $\tilde{I}_j^n \subseteq \mathbb{C}^n$ denote the orthogonal subspace of $\tilde{I}_j^n$. Then, in the regime of asymptotically high transmit powers (i.e., ignoring the noise), the decodability of information symbols at Rx$_j$ corresponds to the constraint that the image of $\text{colspan}(G_j^n V_j^n)$ on $\tilde{I}_j^n$ has dimension $m_j(n)$: $\dim \left( \text{Proj}_{\tilde{I}_j^n} \text{colspan}(G_j^n V_j^n) \right) = \dim \left( \text{colspan}(G_j^n V_j^n) \right) = m_j(n)$, which can be shown by simple linear algebra to be equivalent to the following:

$$\text{rank}(G_j^n [U_{i=1}^k V_j^n]) = \text{rank}(G_j^n [U_{i=1}^k V_j^n]) = m_j(n).$$

(3)

We now define the linear degrees-of-freedom of the $k$-user MISO broadcast channel with hybrid CSIT.

**Definition 2.** $k$-tuple $(d_1, d_2, \ldots, d_k)$ degrees-of-freedom are linearly achievable if there exists a sequence $\{f^{(n)}\}_{n=1}^{\infty}$ such that for each $n$ and the corresponding choice of $(m_1(n), m_2(n), \ldots, m_k(n))$, $(V^n_1, V^n_2, \ldots, V^n_k)$ satisfy the decodability condition of (3) with probability 1, and $d_j = \lim_{n \to \infty} \frac{m_j(n)}{n}$. We also define the linear degrees-of-freedom region $D_{\text{region}}$ as the closure of the set of all achievable $k$-tuples $(d_1, d_2, \ldots, d_k)$. Furthermore, the linear sum-degrees-of-freedom (LDoF$_{\text{sum}}$) is then defined as follows:

$$\text{LDoF}_{\text{sum}} = \max \sum_{j=1}^{k} d_j, \text{ s.t. } (d_1, d_2, \ldots, d_k) \in D_{\text{region}}.$$

(4)
III. Main Result and Its Implications

In this section we state a new outer bound on the LDoF region of the general k-user MISO BC with hybrid CSIT and mention its various implications.

Theorem 1. The following provides an outer bound on the LDoF region $\mathcal{D}_{region}$ for the $k$-user MISO BC with hybrid CSIT:

$$\mathcal{D}_{region} \subseteq \left\{ (d_1, \ldots, d_k) \mid 0 \leq d_1, \ldots, d_k \leq 1, \right. \left. \forall i \in \mathcal{D}, \sum_{j=1}^{\left| \mathcal{P} \right|+\left| \mathcal{D} \right|-1} \frac{d_{i,j} \left| \mathcal{D}_{i,j} \right|}{2^{\left| \mathcal{D}_{i,j} \right|}} + d_i + \sum_{j \in \mathcal{N}} d_j \leq 1 \right\}. \quad (5)$$

$$\forall i \in \mathcal{D}, \sum_{j \in \mathcal{N}} d_j + \sum_{j \in \mathcal{N}} d_j + \sum_{j \in \mathcal{N}} d_j \leq 1 \right\}. \quad (6)$$

Remark 1. The bound in Theorem 1 can be seen as generalization of the bound in [16] for $k = 3$. Theorem 1 extends that result and provides a general outer bound on the LDoF region for $k$-user MISO BC. In particular, the bound in Theorem 1 strictly improves the state-of-the-art bounds, and also leads to complete characterization of $\mathcal{D}_{region}$ for $k = 3$. For instance, for PDD (i.e. Rx) supplying instantaneous CSIT, while Rx2, Rx3 supply delayed CSIT the prior results only suggest that $\text{LDoF}_\text{sum} \leq \frac{k}{2}$ [10], while by using Theorem 1 for $k = 3$, one can show that $\text{LDoF}_\text{sum}$ is indeed equal to $\frac{3}{2}$ [16]. In addition, for PPD, the state-of-the-art bounds suggest that $\text{LDoF}_\text{sum} \leq \frac{7}{3}$, which is a loose bound as $\text{LDoF}_\text{sum} = \frac{3}{2}$ [16].

Theorem 1 enables us to approximately characterize LDoF$_{\text{sum}}$ to within 0.5 for a broad range of CSIT configurations ($|\mathcal{P}| \geq |\mathcal{D}|$), and to characterize LDoF$_{\text{sum}}$ for some special cases ($|\mathcal{D}| = 1$). These results are stated more precisely in the following two Propositions.

Proposition 1. For general $k$-user MISO BC with $|\mathcal{P}| \geq |\mathcal{D}|$,

$$|\mathcal{P}| \leq \text{LDoF}_\text{sum} \leq |\mathcal{P}| + \left| \mathcal{D} \right| \frac{2^{\left| \mathcal{D} \right|}}{2^{\left| \mathcal{P} \right|}+1} = |\mathcal{P}| + \left| \mathcal{D} \right| \frac{2^{\left| \mathcal{D} \right|}}{2^{\left| \mathcal{P} \right|}+1}. \quad (7)$$

Proposition 2. For general $k$-user MISO BC with $|\mathcal{D}| = 1$,

$$\text{LDoF}_\text{sum} = \left| \mathcal{P} \right| + \frac{1}{2^{\left| \mathcal{P} \right|}} \leq \left| \mathcal{P} \right| + \frac{1}{2^{\left| \mathcal{P} \right|}}. \quad (8)$$

Proofs of Propositions 1, 2 are provided in the long version of the paper available on ArXiv [17].

IV. Proof of Theorem 1

In this Section we prove Theorem 1. In particular, we focus on proving (5); proof of (6) is provided in the long version of the paper available on ArXiv [17]. We first provide three main ingredients that result in the proof of Theorem 1, and then show how they are used to prove Theorem 1. The first two ingredients deal with lower bounding received signal dimension at a receiver which supplies delayed CSIT, while the third ingredient captures the impact of no CSIT. The first main ingredient is Interference Decomposition Bound, which is proved in Section V.

Lemma 1. (Interference Decomposition Bound) Consider a fixed linear coding strategy $f^{(n)}$, with corresponding precoding matrices $V_1^n, V_2^n, \ldots, V_k^n$ as defined in (2). For any $S \subseteq \{1, 2, \ldots, k\}$, any $j' \in S$, and any $j \in \{1, 2, \ldots, k\} \setminus S$ for which $I_j = D$, $\text{rank}[V_{j'}^n] + \text{rank}[V_{j'}^n | \cup_{i \neq j'} V_i^n] a.s. \leq \text{rank}[V_{j'}^n | \cup_{i \in S} V_i^n]$. \quad (9)

Lemma 2. (MISO Rank Ratio Inequality) For any linear coding strategy $f^{(n)}$, with corresponding $V_1^n, V_2^n, \ldots, V_k^n$ as defined in (2), and any $S \subseteq \{1, 2, \ldots, k\}$, if $I_j = I_j = D$ for some $j, j' \in \{1, 2, \ldots, k\}$, then

$$\text{rank}[V_{j'}^n | \cup_{i \in S} V_i^n] a.s. \leq 2 \times \text{rank}[V_{j'}^n | \cup_{i \in S} V_i^n]. \quad (10)$$

Remark 3. Also, note that Lemma 2 is the MISO version of the Rank Ratio Inequality in [4], [15].

The third main ingredient, named Least Alignment Lemma, demonstrates that when using linear schemes, once the transmitter has no CSIT with respect to a certain receiver, the least amount of alignment will occur at that receiver, meaning that transmit signals will occupy the maximal signal dimensions at that receiver. Proof of the Lemma follows the same steps as in [12]; so it is omitted for brevity.

Lemma 3. (Least Alignment Lemma) For any linear coding strategy $f^{(n)}$, with corresponding $V_1^n, V_2^n, \ldots, V_k^n$ as defined in (2), and any $S \subseteq \{1, 2, \ldots, k\}$, if $I_j = N$ for some $j \in \{1, 2, \ldots, k\}$, then for all $j' \in \{1, 2, \ldots, k\}$,

$$\text{rank}[V_{j'}^n | \cup_{i \in S} V_i^n] a.s. \leq \text{rank}[V_{j'}^n | \cup_{i \in S} V_i^n]. \quad (11)$$

Remark 4. The statement of Lemma 3 does not assume any specific CSIT with respect to any receiver except Rxj.
Claim 1. We upper bound each term on the L.H.S. of (12) separately. where (a) follows from sub-modularity of rank; and (b) follows
\[ \forall \pi \in \mathcal{N} \leq |D| \]
which is the third term on the L.H.S. of (12). By (11), for all
\[ \sum_{j \in \pi} \frac{m_j(n)}{2^j} + m_{|P|+|D|}(n) + \sum_{j \in \mathcal{N}} m_j(n) \leq n. \]  
(12)
We upper bound each term on the L.H.S. of (12) separately. The following claim, proved in [17], provides an upper bound on the first term on the L.H.S. of (12).

Claim 1. \[ \sum_{j \in \pi} \frac{m_j(n)}{2^j} \leq \text{rank}[G_k^{n_1}[V^{n_1}_{j=1} V^n_i]] - \text{rank}[G_k^{n_1}[U_j \neq |P|+|D|] V^n_i]. \]  
(13)
We now upper bound \[ m_{|P|+|D|}(n) \] which is the second term on the L.H.S. of (12). By (11) we obtain
\[ m_{|P|+|D|}(n) \leq \text{rank}[G_k^{n_1}[P^{n_1}_i] V^n_i] \]
\[ \leq \text{rank}[G_k^{n_1}[P^{n_1}_i] V^n_i]. \]  
(14)
where (a) follows from sub-modularity of rank; and (b) follows by Least Alignment Lemma (Lemma 3) since receiver \[ |P| + |D| + 1 \] supplies no CSIT. We now upper bound \[ \sum_{j \in \pi} \frac{m_j(n)}{2^j} \] which is the third term on the L.H.S. of (12). By (11), for all
\[ \sum_{j \in \mathcal{N}} m_j(n) \leq \text{rank}[G_k^{n_1}[V^n_i \ldots V^n_i]] - \text{rank}[G_k^{n_1}[U_j \neq |P|+|D|] V^n_i]. \]  
(15)
Note that since receivers with index in \{ |P| + |D| + 1, \ldots, k \} supply no CSIT, and due to their symmetry, for each \[ i \in \{ |P| + |D| + 1, \ldots, k \} \] we have
\[ \text{rank}[G_k^{n_1}[V^n_i \ldots V^n_i]] \leq \text{rank}[G_k^{n_1}[V^n_i \ldots V^n_i]]. \]  
(16)
Therefore, by (15), (16) we obtain
\[ \sum_{j \in \mathcal{N}} m_j(n) \leq \text{rank}[G_k^{n_1}[V^n_i \ldots V^n_i]] - \text{rank}[G_k^{n_1}[V^n_i \ldots V^n_i]]. \]  
(17)
Hence, by summing the inequalities in (13), (14), and (17),
\[ \sum_{j \in \mathcal{N}} m_j(n) \leq \text{rank}[G_k^{n_1}[V^n_i \ldots V^n_i]] - \text{rank}[G_k^{n_1}[V^n_i \ldots V^n_i]]. \]  
(18)
which proves (12), thus, proving bound (5) in Theorem 1.

V. PROOF OF INTERFERENCE DECOMPOSITION BOUND
Consider a fixed linear encoding function \( f(n) \), with corresponding precoding matrices \( V^n_1, \ldots, V^n_k \) as defined in (2).

Definition 3. For any \( S \subseteq \{ 1, \ldots, k \}, j' \in S, j \in \{ 1, \ldots, k \} \),
\[ \mathcal{T}_1 = \{ t \in \{ 1, \ldots, n \} \mid \text{rank}[G_j^{n_1}[U_{i \in S} V^n_i]] = \text{rank}[G_j^{n_1-1}[U_{i \in S} V^n_i]] + 1 \} \]
\[ \mathcal{T}_2 = \{ t \in \mathcal{T}_1 \mid \text{span}[G_j^{n_1-1}[U_{i \in S} V^n_i]] \subseteq \text{range}[G_j^{n_1}[U_{i \in S} V^n_i]]. \}
\]
Remark 5. \( \mathcal{T}_1 \) is the subset of timeslots in which received signal dimension at \( R_{j'} \) increases, while \( \mathcal{T}_2 \) is the subset of \( \mathcal{T}_1 \) in which the received signal at \( R_{j'} \) is already recoverable by using the past received signals at \( R_j \). The definitions of \( \mathcal{T}_1, \mathcal{T}_2 \) focus only on the contribution of \( V^n_i \), where \( i \in S \), on the dimension of received signals at different receivers; since the statement of Lemma 1 only involves \( V^n_i \), where \( i \in S \).

We now state two lemmas that are the main building blocks of the proof of Lemma 1.

Lemma 4. \( \text{rank}[G_j^{n_1}[V^n_i]] - (|\mathcal{T}_2| - |\text{rank}[G_j^{n_1}[U_{i \in S} V^n_i]]|) \leq \text{rank}[G_j^{n_1}[U_{i \in S} V^n_i]]. \]

Lemma 5. \( |\mathcal{T}_2| - |\text{rank}[G_j^{n_1}[U_{i \in S} V^n_i]]| \leq \text{rank}[G_j^{n_1}[U_{i \in S} V^n_i]]. \)

Proof of Lemma 1 follows immediately by summing the inequalities in Lemma 4, 5. Moreover, the proof of Lemma 5 follows from straightforward linear algebra, and can be found in [17]. Hence, we will only prove Lemma 4 here.

Let us denote the indicator function by \( I(\cdot) \). We then have
\[ n - \text{rank}[G_j^{n_1}[U_{i \in S} V^n_i]] \]
\[
\sum_{t=1}^{n} I(\text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}]) = \text{rank}[G_{t}^{j-1} | \bigcup_{u \in S} V_{u}^{t-1}])
\]
\[
\sum_{t \in T} I(\text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}]) = \text{rank}[G_{t}^{j-1} | \bigcup_{u \in S} V_{u}^{t-1}])
\]
\[
+ \sum_{t \in T} I(\text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}]) = \text{rank}[G_{t}^{j-1} | \bigcup_{u \in S} V_{u}^{t-1}])
\]
\[
\sum_{t \in T} I([\bigcup_{u \in S} V_{u}(t)] \in \text{rowspan}[G_{t}^{j-1} | \bigcup_{u \in S} V_{u}^{t-1}])
\]
\[
+ \sum_{t \in T} I(\text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}]) = \text{rank}[G_{t}^{j-1} | \bigcup_{u \in S} V_{u}^{t-1}])
\]
\[
\leq \sum_{t \in T} I(\text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}]) \leq \sum_{t \in T} I([\bigcup_{u \in S} V_{u}(t)] \in \text{rowspan}[G_{t}^{j-1} | \bigcup_{u \in S} V_{u}^{t-1}])
\]
\[
|T| + n - \text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}])
\]
\[
= |T| + n - \text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}])
\]
\[
- (\text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}])
\]
\[
(\text{Sub-modularity})
\]
\[
\leq |T| + n - \text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}])
\]
\[
- (\text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}])
\]
\[
(a.s.)
\]
\[
|T| + n - \text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}])
\]
\[
- (\text{rank}[G_{t}^{j} | \bigcup_{u \in S} V_{u}^{t}])
\]
\[
(3)
\]

where (a) is due to Lemma 10 in [17]; (b) follows immediately from the definition of \(T\). By rearranging the above inequality the proof of Lemma 4 will be complete.

VI. CONCLUSION

In this paper we studied the impact of heterogeneous CSIT on the linear DoF (LDoF) of multi-antenna multi-user networks in the context of \(k\)-user MISO broadcast channel with hybrid CSIT, where the CSIT supplied by each receiver can be instantaneous, delayed, or none. We introduced a new outer bound which leads to exact characterization of LDoF for specific cases, and an approximate characterization of LDoF for scenarios with more receivers supplying instantaneous CSIT than receivers supplying delayed CSIT. For other scenarios, improving upon the current results would be an interesting direction.

Another interesting future direction is to extend the results to the non-linear setting (DoF). To this aim, one needs to extend the three proof ingredients to the non-linear setting. Entropy version of Lemma 2 has been proved in prior works [18]; and Least Alignment Lemma has recently been extended to the non-linear setting in [19]. Hence, an interesting direction is to extend the Interference Decomposition Bound to the non-linear setting.

VII. ACKNOWLEDGMENT

The research of A.S. Avestimehr and S. Lashgari is supported by NSF Grants CAREER 1408639, CCF-1408755, NETS-1419632, EARS-1411244, ONR award N000141310094, and research grants from Intel and Verizon via the 5G project. The work of R. Tandon is supported by the NSF Grant CCF-1442090.

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