Chap. 6 & Chap. 12: Performability Modeling

6.4 Markov Reward Model

We can associate each state with a "reward" denoting the performance level given by the system while it is in that state.

Markov process: $X = \{X(t),\ t \ge 0\}$

State Probabilities: $\pi_j(t) = \Pr(X(t) = j),\ \underbrace{j \in S}$

Steady-state probabilities: $\pi_j,\ j \in S$

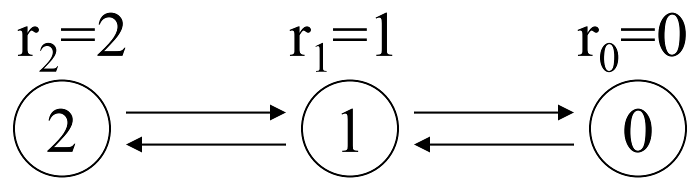
State j is associated with a reward r_j

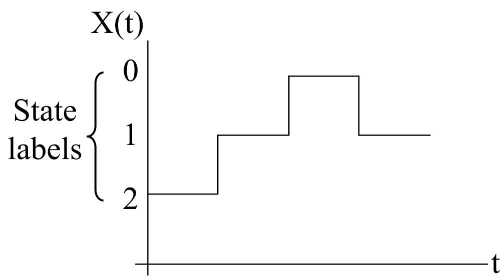
Let $Z(t) = r_{\underbrace{X(t)}}$ be the system reward at time t

denoting which state the system is in at time t in the markov process

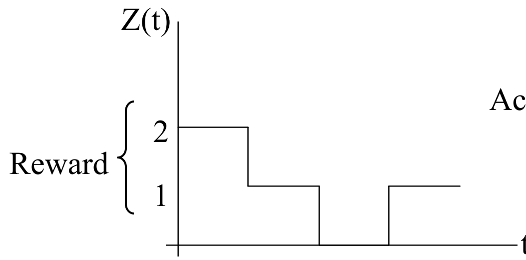
Then, the amount of reward accumulated during an interval (0, t) is given by: $Y(t) = \int_0^t Z(t)\,dt$

Ex:

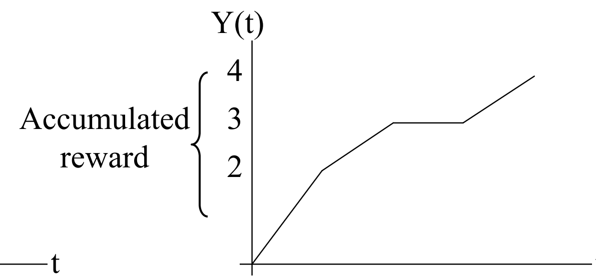




X(t) vs. t

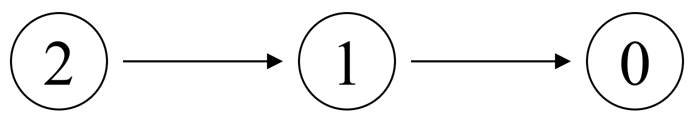


Z(t) vs. t



Y(t) vs. t

$$\lim_{t\to\infty} Y(t) = ? \quad \begin{cases} \text{with no absorbing states: } \infty \text{ (not defined)} \\ \text{with absorbing states: a finite value, denoted by} \\ \quad Y(\infty)\text{: } \underline{\text{the accumulated reward until absorption}} \end{cases}$$



An absorbing state

Performability Measures:

a) Expected reward at time t

$$E[Z(t)] = \sum_{i \in S} r_i \cdot \pi_i(t)$$

can be used to represent the instantaneous "computational capacity" of the system at time t

Sharpe:
exrt(t; system-name)

b) Expected reward at steady state

$$E[Z(t=\infty)] = \sum_{i \in S} r_i \cdot \pi_i$$

Sharpe:
exrss(system-name)

Meaningless for a Markov chain with absorbing states (i.e., meaningful only for irreducible Markov models which by definition do not have absorbing states)

c) Expected cumulative reward over the interval [0, t]

$$E[Y(t)] = E\left[\int_0^t Z(t)\,dt\right] = \int_0^t E[Z(t)]\,dt$$

Sharpe:
cexrt(t; system-name)

$$= \int_0^t \sum_{i\in S} r_i \cdot \pi_i(t)\,dt = \sum_{i\in S} r_i \underbrace{\int_0^t \pi_i(t)\,dt}$$

$$= \sum_{i\in S} r_i L_i(t)$$

Expected total time that the Markov chain stays at state i during the time interval [0, t]

d) Time averaged cumulative reward

W(t) = Y(t) / t

with absorbing states:

Y(∞) is finite ∴ as t → ∞, W(∞) = 0

with no absorbing states: W(∞) is finite

$$\boxed{\text{sharpe : } \frac{cexrt(t;\text{system name})}{t} \text{ for } \frac{E[Y(t)]}{t}}$$

e) Distribution of cumulative reward:

(a hard problem)

$$\underbrace{cdf_{Y(t)}}_{\underline{prob\{Y(t)\leq r\}}} \text{ or } pdf_{Y(t)}$$

Sharpe: not provided

Usage: can answer the following question:

<u>What is the probability that the system is able to achieve a given amount of work *r* during the interval [0, t]?</u>

f) Probability that the "**cumulative reward until absorption**" $Y(\infty)$ is less than or equal to *r* when an absorbing state is reached:

$$prob\{Y(\infty) \leq r\}$$

$\Rightarrow$ meaningful only for a Markov model with absorbing states

Sharpe: $\underbrace{\text{reward(system-name)}}_{\text{(in symbolic form)}}$ or $\underbrace{\text{rvalue(r;system-name)}}_{\text{(in numerical form)}} = prob\{Y(\infty)\leq r\}$

Reward assignments

Ex1: An irreducible Markov model (no absorbing states)

reward assignment: $r_i = 1$ to operational states

$r_i = 0$ to non-operational states

$E[Y(\infty)]$ infinity (undefined)

$E[Z(t)]=A(t)$ *availability at time t*

$E[Z(\infty)]=E[Z]=A$ *steady state availability*

$E[Y(t)]=$ *expected system up time during [0,t]*

Same assignments for a Markov chain with absorbing states?

$E[Z(\infty)]=0$

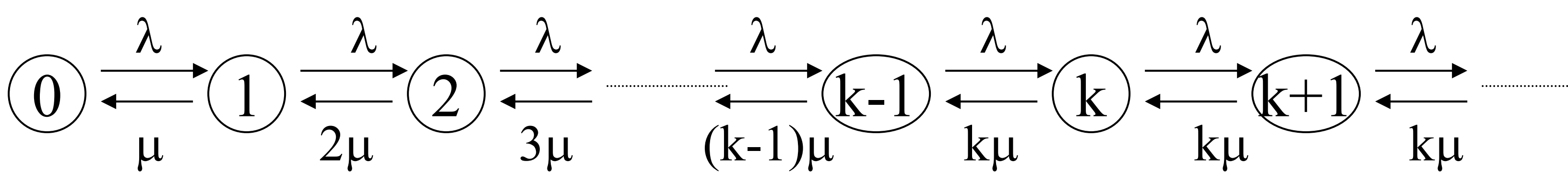
$E[Y(t)]=\int_0^t R(t)dt$

$E[Z(t)]=R(t)$ *reliability*

$E[Y(\infty)]=MTTF=\int_0^\infty R(t)dt$

k servers

Ex 2: A birth-death process modeling an M/M/k



Suppose we know the s.s. probability vector $\pi = \{\pi_0, \pi_1, \pi_2 \ldots\}$

1) assign a reward of "# of customers" to each state

$E[Z(t)]$ = *expected population at time t*

$E[Z(\infty)] = E[Z]$ = *steady state population*

2) assign a reward of "service rate" to each state

$E[Z(t)]$ = *expected throughput at time t*

$E[Z(\infty)]$ = *steady state throughput*

Ex 3: 2P3M without repair capability

the system functions if at least $\overbrace{\text{1 processor}}^{\lambda_p}$ & $\overbrace{\text{1 mm}}^{\lambda_m\text{:failure rate}}$ functioning

P.314
chap.12

- state representation: $(\overset{\text{mm}}{i}, \overset{\text{CPU}}{j})$
- assume that the service rate of the system in state (i, j) is:

$$r_{ij} = m\left(1 - \left(1 - \frac{1}{m}\right)^l\right) \quad \text{when } l = \min(i, j) \text{ and } m = \max(i, j)$$

1) assign a reward of "service rate" to each state

$E[Z(t)]$ = *expected throughput at time t*

$E[Y(\infty)]$ = *expected # of customers serviced before failure*

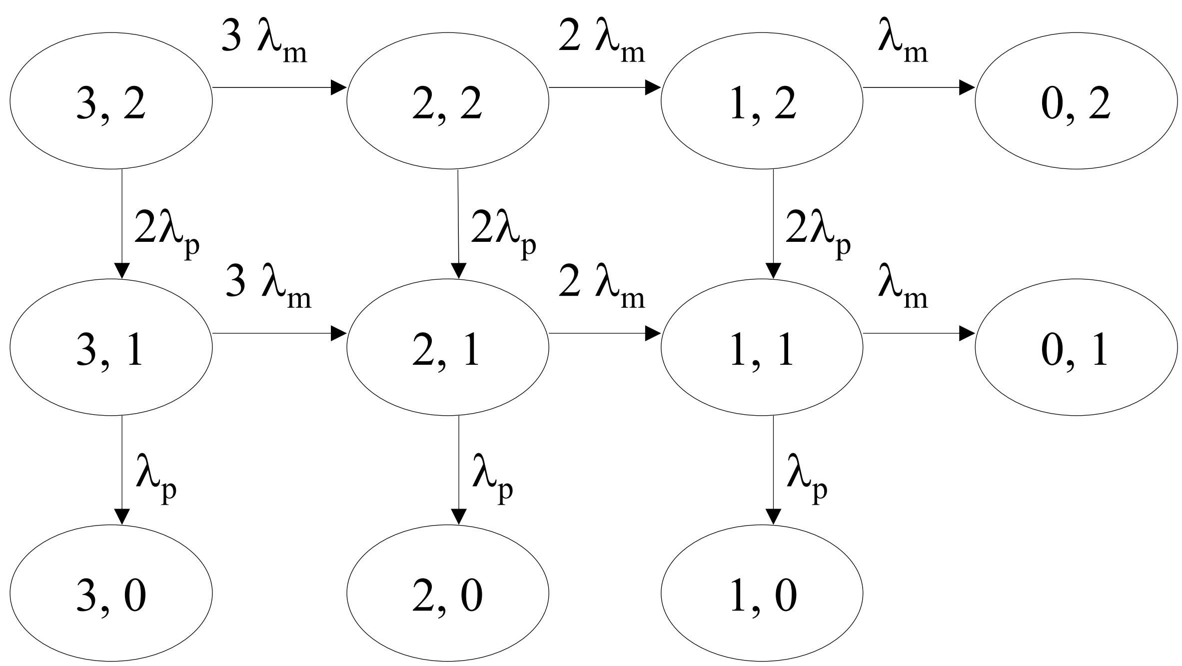
2) reward assignment: $r_i = 1$ to operational states, i.e., (3,2), (2,2) (1,2), (3,1), (2,1) and (1,1)

$r_i = 0$ to non-operational states

$E[Z(t)]$ = *reliability at time t*

$E[Y(\infty)]$ = *MTTF*

A Markov chain for reliability analysis of a system without repair capability



- $\lambda_p = 1/(2*720)$, $\lambda_m = 1/(720)$

```
bind    λm    1/(2*720)
bind    λp    1/720
Markov  3mem-2proc
* memory failure
  32   22   3* λm
  22   12   2* λm
  12   02   λm
  31   21   3*λm
  21   11   2*λm
  11   01   λm
* processor failure
  32   31   2* λp
  31   30   λp
  22   21   2*λp
  21   20   λp
  12   11   2*λp
  11   10   λp
* reward assignment
  reward
  32   r32
  22   r22
  12   r12
  31   r31
  21   r21
  11   r11
* default is 0 assigned to other states
  end
  32   1.0
  end
```

expected reward (throughput) at time t=20

P.375 sum(index,low,high, expression)

```
*
* Reward assignment is the
* service rate in state (i, j)
bind
   r32   15/9
   r22   3/2
   r12   1
   r31   1
   r21   1
   r11   1
end
```

Probability of the system serving less than 200 customers before it fails

```
* print prob{Y(∞) ≤ r}
* in symbolic form
reward (3mem-2proc)
* print prob{Y(∞) ≤ 200}
rvalue (200; 3mem-2proc)
* print E[Z(20)]
exrt (20; 3mem-2proc)
* print E[Z(20)] again based on
* the definition of E[Z(t)], i.e.,
*    E[Z(t)] = Σ_ij r_ij * π_ij(t)
expr sum(i, 1, 3, sum(j, 1, 2,\
(sreward(3mem-2proc, $(i)$(j))*\
value(20;3mem-2proc, $(i)$(j)))))
*
* sreward returns the reward assigned
* to a state
*
* Reward assignment to calculate R(t)
*
bind   r32   1
   :     :
bind   r11   1
*
* print R(t) at t=20
expr exrt(20; 3mem-2proc)
* code to be continued in the next page
```

$$E[Z(t)] = \sum_{ij} r_{ij} * \pi_{ij}(t)$$

* R(20) is the same as E[Z(20)] with this reward assignment

```
expr    exrt (20; 3mem-2proc)
expr    value (20; 3mem-2proc, 32) + \
        value (20; 3mem-2proc, 31) + \
        value (20; 3mem-2proc, 22) + \
        value (20; 3mem-2proc, 21) + \
        value (20; 3mem-2proc, 12) + \
        value (20; 3mem-2proc, 11)
```

These two should give the same result

* compare E[Z(t)], E[Y(t)] & $\frac{E[Y(t)]}{t}$

* we expect $E[Z(t)] \approx \frac{E[Y(t)]}{t}$ as t increases

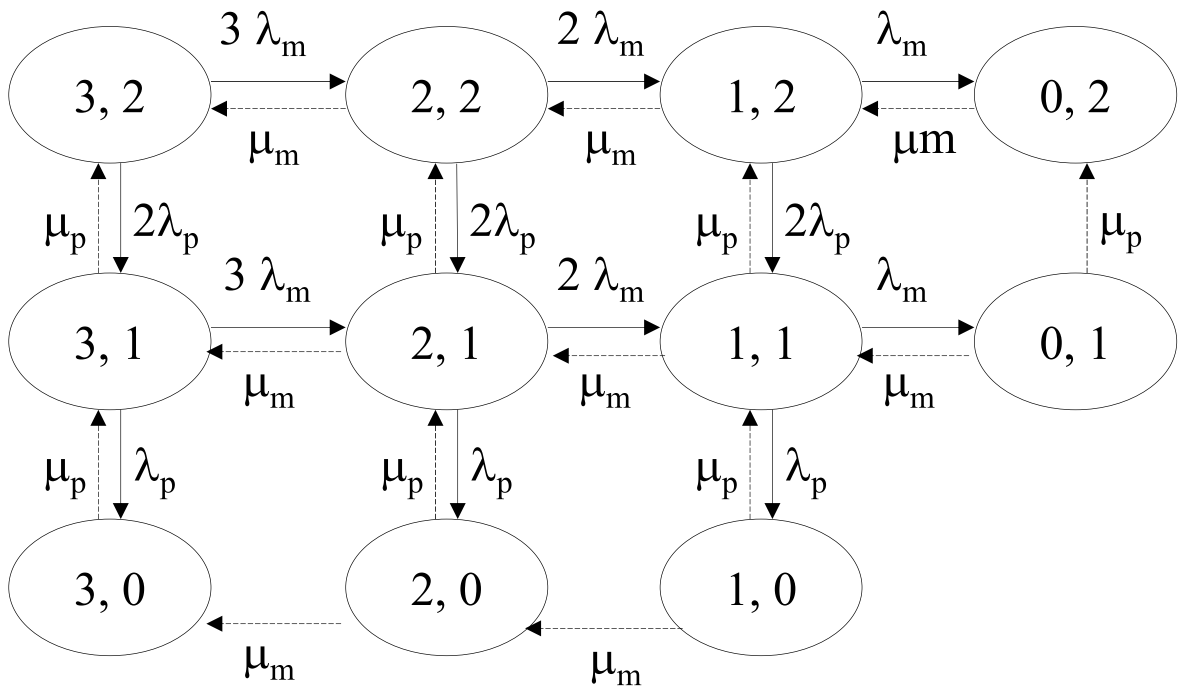
```
loop    t, 0, 30, 5
            expr    exrt (t; 3mem-2proc)
            expr    cexrt (t; 3mem-2proc)
            expr    cexrt (t; 3mem-2proc)/t
end
```

```
*
* What is E[Y(∞)] with
* this reward assignment?
*
```

* end the sharpe program

end

An <u>acyclic</u> (irreducible) Markov chain for availability analysis



P. 318:
This Markov model is irreducible due to repair

- Per processor λ_p= 1/(2*720), per MM λ_m= 1/(720)
- Once a system enters a failure state, the system halts until it enters an operational state again via repair
- There is 1 repair facility for processors with the repair rate of μ_p = 1/4 & 1 repair facility for memory modules with the repair rate of μ_m = 1/2, so simultaneous repair is possible in this case.

```
bind  λm   1/(2*720)
bind  λp   1/720
bind  μm   1/2
bind  μp   1/4
Markov 3mem-2proc   readprobs
* memory failure
  32  22  3* λm
  22  12  2* λm
  12  02  λm
  31  21  3*λm
  21  11  2*λm
  11  01  λm
* processor failure
  32  31  2* λp
  31  30  λp
  22  21  2*λp
  21  20  λp
  12  11  2*λp
  11  10  λp
* reward assignment
  reward
  32  r32
  22  r22
  12  r12
  31  r31
  21  r21
  11  r11
* default is 0 assigned to other states
  end
  32  1.0
  end
```

required for transient analysis of irreducible Markov chain

```
* processor repair
  30  31  up
  31  32  up
  20  21  up
  21  22  up
  10  11  up
  11  12  up
  01  02  up
* memory repair
  22  32  μm
  12  22  μm
  02  12  μm
  21  31  μm
  11  21  μm
  01  11  μm
```

expected reward (throughput) at time t=20

```
* Reward assignment is the
* service rate in state (i, j)
bind
   r32  15/9
   r22  3/2
   r12  1
   r31  1
   r21  1
   r11  1
end
```

```
* prob{Y(∞) ≤ r}
* is not meaningful in this case
* Print expected cumulative # of clients
* served at t=50 as E[Y(50)]
  cexrt (50; 3mem-2proc)
* print E[Z(20)]
  exrt (20; 3mem-2proc)
* print E[Z(20)] again based on
* the definition of E[Z(t)], i.e.,
*
```

$$E[Z(t)] = \sum_{ij} r_{ij} * \pi_{ij}(t)$$

```
expr sum(i, 1, 3, sum(j, 1, 2\
(sreward(3mem-2proc, $(i)$(j))*\
value(20;3mem-2proc, $(i)$(j)))))
* reward assignment for availability
bind   r32   1
   :     :
bind   r11   1
*
* print A(t) at t=20 as E[Z(t=20)];
expr exrt(20; 3mem-2proc)
*
* When repair exists & Markov chain is
* irreducible, E[Z(∞)] exists; it is
* the steady state availability in this case
expr exrss (3mem-2proc)
end
```