Chap 2: Reliability and Availability Models

Reliability

$R(t)$ = prob{S is fully functioning in [0,t]}

Suppose from [0,t] time period, we measure out of N components, of which

$$\begin{cases} N_0(t)\text{: \# of components operating correctly at time t} \\ N_f(t)\text{: \# of components which have failed at time t} \end{cases}$$

$$R(t) = \text{reliability of the component}$$

$$= \frac{N_0(t)}{N} = \frac{N - N_f(t)}{N} = 1 - \frac{N_f(t)}{N}$$

$$\frac{dR(t)}{dt} = \frac{-1}{N}\frac{dN_f(t)}{dt}$$

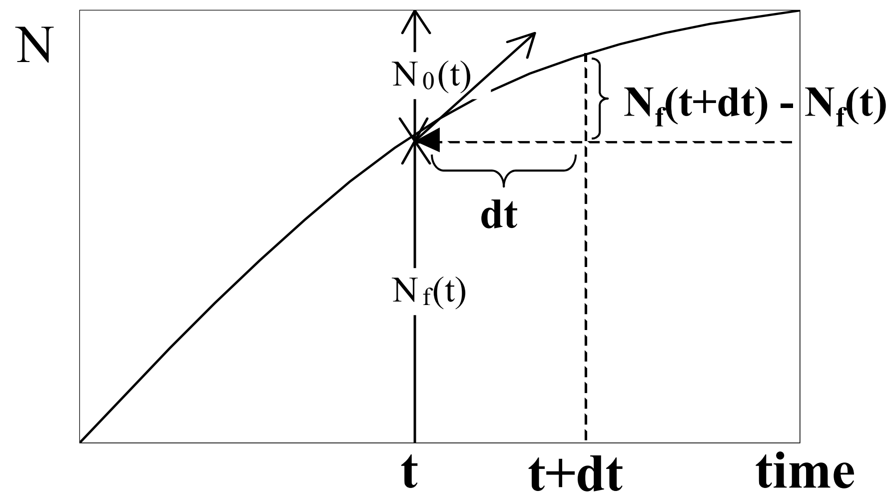
$$\underbrace{\frac{dN_f(t)}{dt}} = -N\,\frac{dR(t)}{dt}$$

Physical meaning:
instantaneous rate at which components are failing

How many unfailing components are there at time t ? $\Rightarrow N_0(t)$

$$\therefore \underbrace{Z(t)} * N_0(t) = \frac{dN_f(t)}{dt}$$

Failing rate of
a single component



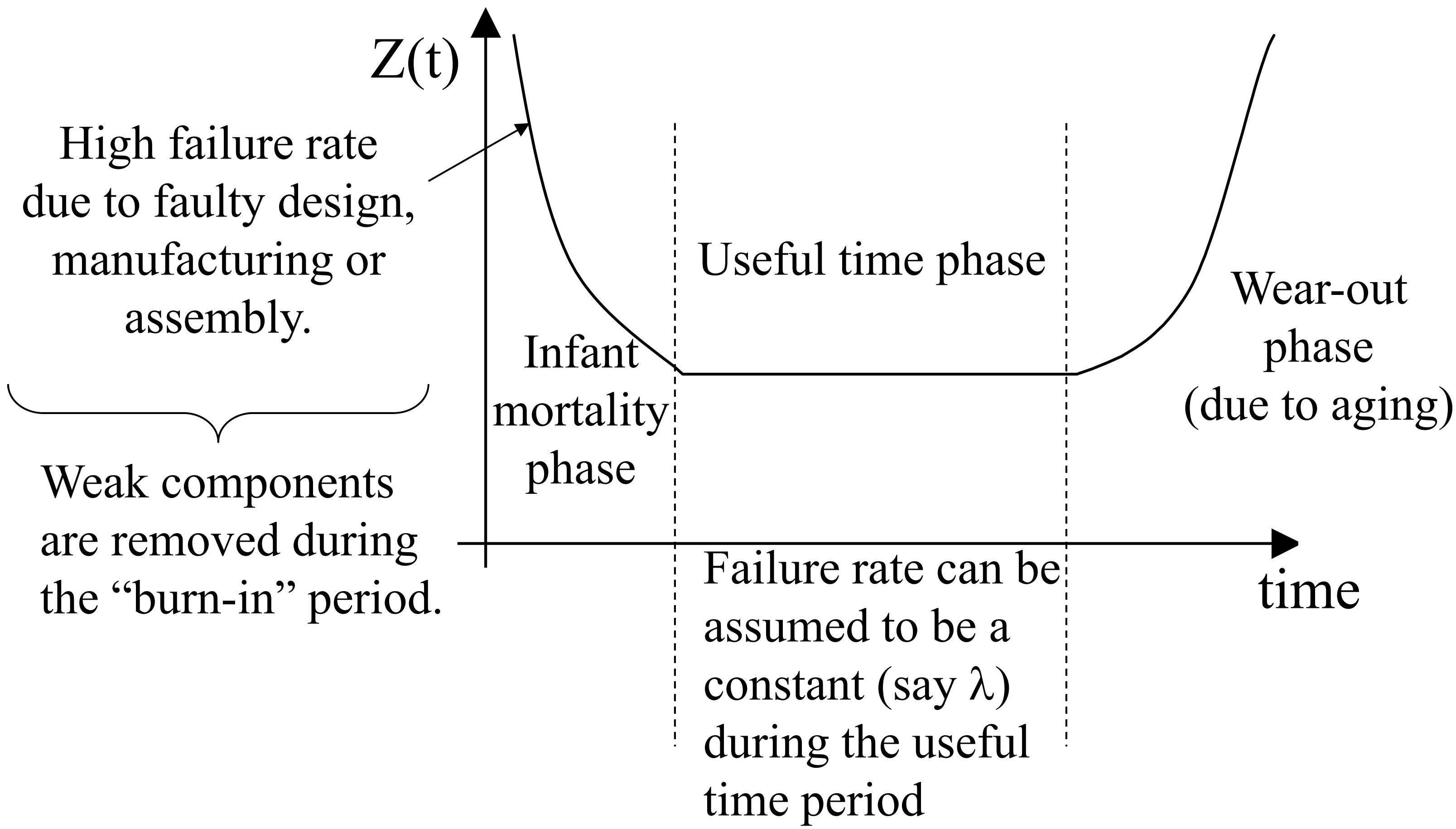
$$Z(t) = \frac{1}{N_0(t)} \frac{dN_f(t)}{dt}$$

$$= \frac{1}{N_0(t)} \left\{ -N \frac{dR(t)}{dt} \right\} = \frac{-\frac{dR(t)}{dt}}{R(t)} = \frac{f(t)}{R(t)} \qquad (2.3)$$

text p.28

$Z(t)$ is also called the <u>hazard rate.</u>

For electronic components, Z(t)'s relationship with respect to time is a bathtub curve.



$$\lambda = \frac{-1}{R(t)} \frac{dR(t)}{dt}$$

$$\frac{dR(t)}{dt} = -\lambda\, R(t)$$

Exponential Failure Law $\boxed{R(t) = e^{-\lambda t}}$

e.g. $\lambda = 0.01\ hr^{-1}$, what is R(t) at t = 100 hrs? Ans: $e^{-0.01*100}$

* For hardware components, exponential failure law is frequently assumed.

* For software components, the reliability may grow as the software's design faults are removed during the testing/debugging phase.

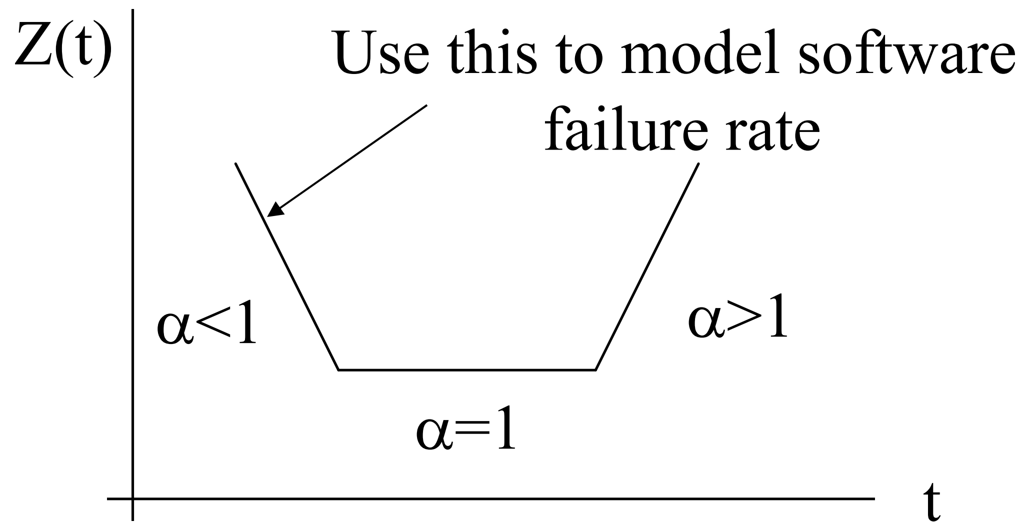
In general, we can assume

$$Z(t) = \alpha\lambda\,(\lambda\, t\,)^{\alpha-1} \qquad \text{Weibull dist.}$$

$$\Downarrow$$

$$R(t) = e^{-(\lambda t\,)^{\alpha}}$$

This yields



e.g. $\alpha = -1$

$$R(t) = e^{\frac{-1}{\lambda t}}$$

$$\therefore\ t \to \infty,\ R(t) \to 1$$
$$t \to 0,\ R(t) \to 0$$

$\therefore$ Reliability improves as a function of t

Formal Definition of *R*(t):

Let x be a γ.ν. representing the life of a system and let F be the cumulative distribution function (CDF) of x.

Then,

$$R(t) = pr\{x > t\} = 1 - F(t) = \int_t^{\infty} f(x)dx$$

∴ For a component obeying the exponential failure law

$$R(t) = \int_t^{\infty} \lambda e^{-\lambda x}\, dx = e^{-\lambda t}$$

Mean Time to Failure (MTTF):

i	first failing time
1	t_1
Identical systems	
N	t_N

The expected time that a system will operate before the first failure occurs

Discrete case

$$MTTF = \frac{\sum_i t_i}{N}$$

Continuous case

failure time

$$MTTF = E[T] = \int_0^\infty t\, f(t)dt = \int_0^\infty t \frac{dF(t)}{dt} dt = \int_0^\infty t \cdot \frac{d(1-R(t))}{dt} dt$$

$$= -\int_0^\infty t \cdot \frac{dR(t)}{dt} dt = \left[-tR(t) + \int 1 \cdot R(t)dt \right]_0^\infty$$

$$= \int_0^\infty R(t)dt \qquad \because R(t) \to 0 \text{ as } t \to \infty$$

$$e.g. \quad MTTF = \int_0^\infty e^{-\lambda t}\, dt = \frac{1}{\lambda}$$

$$\int uv'dt = uv - \int u'vdt$$

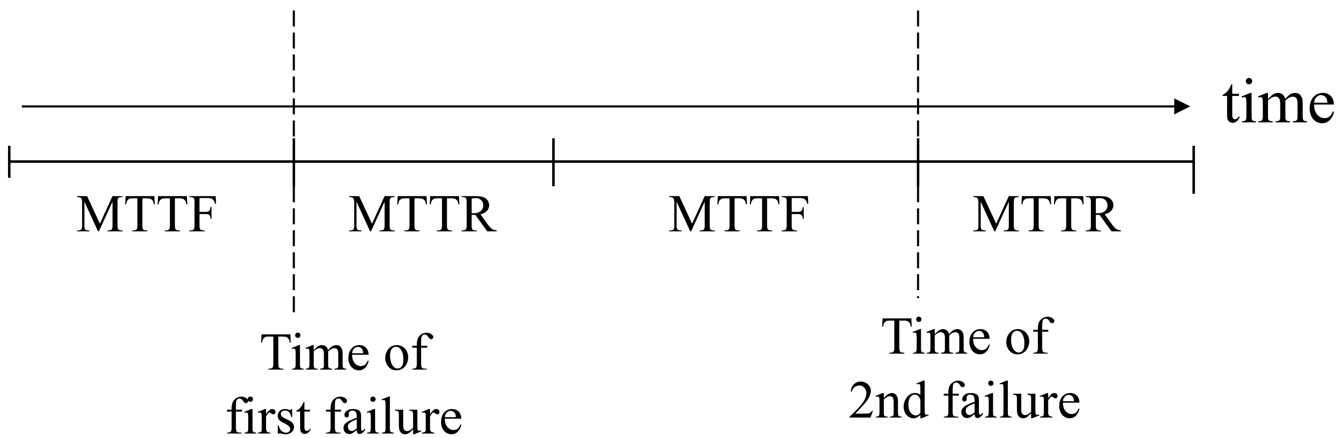
Q: what is the reliability of a system obeying the exponential failure law at t = MTTF?

Ans: $R(t = MTTF) = R\left(t = \frac{1}{\lambda}\right) = e^{-\lambda\left(\frac{1}{\lambda}\right)} = e^{-1} = \frac{1}{e} = \frac{1}{2.718\ldots} = 0.3678\ldots$

MTTR (Mean Time to Repair)

If we also assume a failed system obeys "Exponential Repair Law", then $MTTR = \frac{1}{\mu}$, where μ is the repair rate

Relationship between MTBF (Mean time between failure), MTTR & MTTF:



$$\boxed{MTBF = MTTR + MTTF}$$

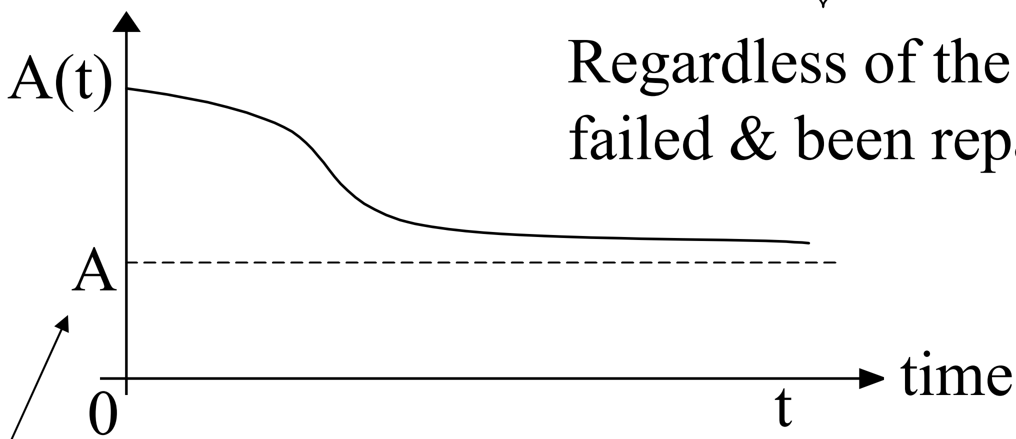
∴ If MTTF >> MTTR Then MTBF ≈ MTTF

Availability

Instantaneous (or point) Availability

$A(t) = \text{prob}\ \{\text{the system is functioning at time } t\}$

Regardless of the # of times it may have failed & been repaired during [0, t]



Steady-State Availability = $\dfrac{MTTF}{MTTF+MTTR}$

$*$ For a system without repair, $A(t) = R(t)$

Assume exponential "failure" & "repair" law

$$O \underset{\mu}{\overset{\lambda}{\rightleftarrows}} F$$

See also page 67, text chapter 4

<u>Time domain:</u>

$$P_o'(t) = -\lambda P_o(t) + \mu P_F(t)$$

$$P_F'(t) = \lambda P_o(t) - \mu P_F(t)$$

with initial state $P_o(0) = 1$ & $P_F(0) = 0$

<u>Laplace domain:</u>

$$SP_o(S) - \overbrace{1}^{P_o(0)} = -\lambda P_o(S) + \mu P_F(S)$$

$$SP_F(S) - \underbrace{0}_{P_F(0)} = \lambda P_o(S) - \mu P_F(S)$$

$$\therefore P_o(S) = \frac{1}{S+(\lambda+\mu)} + \frac{\mu}{S(S+\lambda+\mu)} = \frac{A}{S} + \frac{B}{S+(\lambda+\mu)}$$

$$= \frac{\frac{\mu}{\mu+\lambda}}{S} + \frac{\frac{\lambda}{\lambda+\mu}}{S+(\lambda+\mu)}$$

Similarly

$$P_F(S) = \frac{\frac{\lambda}{\lambda+\mu}}{S} - \frac{\frac{\lambda}{\lambda+\mu}}{S+(\lambda+\mu)}$$

Inverse LT to return to time domain

$$F(t) \underset{\text{Inverse LT}}{\overset{\text{LT}}{\rightleftarrows}} \mathcal{L}(F(t)) = f(S)$$

1	$1/S$
t	$1/S^2$
t^n	$n!/S^{n+1}$
e^{at}	$1/(S-a)$

$$P_o(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$P_F(t) = \frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

Physical meaning:

$P_o(t)$ = prob {the system is operational at time t}

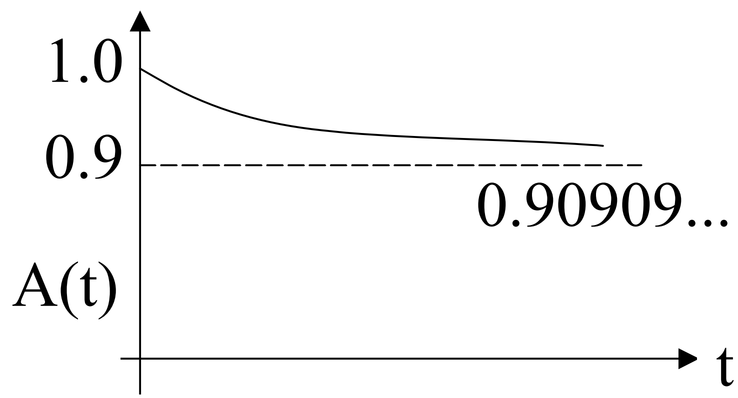
$$\therefore A(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

Q1: unavailability? Ans: $P_F(t)$

Q2: R(t)? Still $e^{-\lambda t}$

By solving $O \xrightarrow{\lambda} F$

e.g. $\lambda = 0.01$ & $\mu = 0.1$



Q3: Steady-state availability?

Ans: $A(t \to \infty) = \dfrac{\mu}{\mu+\lambda}$

$$= \frac{MTTF}{MTTF + MTTR} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{1}{\mu}} = \frac{\mu}{\mu + \lambda}$$

Modeling:

Series-Parallel Reliability Block Diagrams

∗ A series-parallel block diagram represents the logical structure of a system with regard to how the reliabilities of its components affect the system reliability.

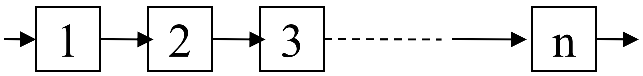
∗ Components are combined into blocks in

— series

— parallel

or — k-out-of-n configurations

A. Serial system: each element of the system is required to function correctly for the system to function correctly.



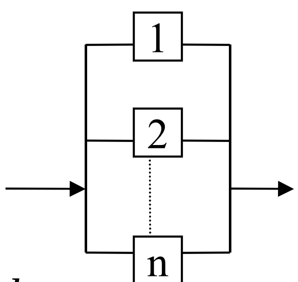
$$\boxed{R_{series}(t) = \prod_{i=1}^{n} R_i(t)} \quad 1$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \ldots e^{-\lambda_n t} = e^{-(\lambda_1 + \ldots + \lambda_n)t}$$

$$= e^{-\lambda t}$$

$$\lambda_{series} = \sum_{i=1}^{n} \lambda_i$$

B. Parallel system: only one of several elements must be operational for the system to be operational.



Assumptions:
- *independent random variables*
- *perfect coverage so up to n-1 failures can be tolerated.*

$$R_{parallel}(t) + F_{parallel}(t) = 1$$

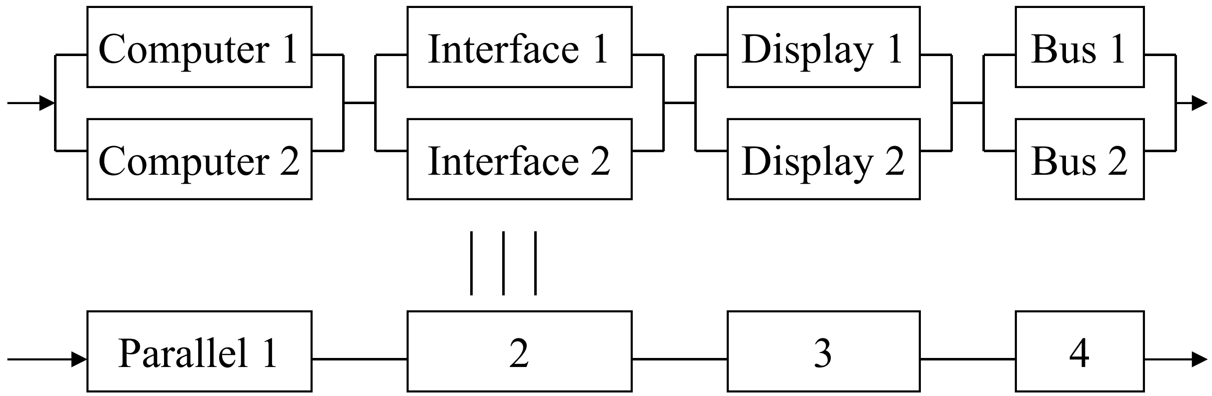
$$\therefore R_{parallel}(t) = 1 - F_{parallel}(t) = 1 - \prod_{i=1}^{n} F_i(t)$$

$$\boxed{R_{parallel}(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t))} \quad 2$$

$$ex: \; R(t) = 1 - \prod_{i=1}^{n} \left(1 - e^{-\lambda_i t}\right)$$

C. Combination of series & parallel systems

e.g.,



$$R_{series} = \prod_{i=1}^{4} R_{parallel, i}$$

$$= \prod_{i=1}^{4}\left(1 - \prod_{j=1}^{2}(1 - R_{j, i})\right)$$

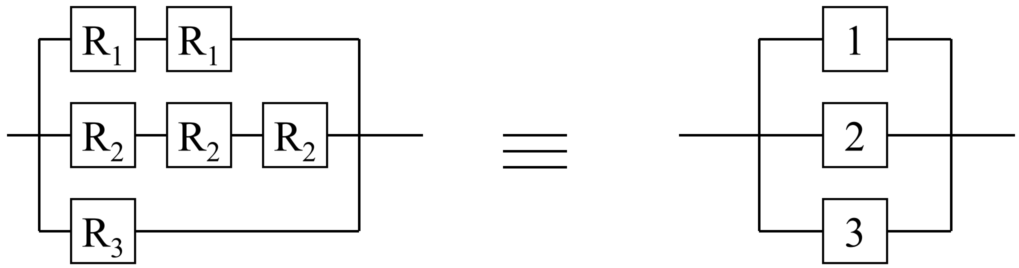
where $R_{j,i}$ is the reliability of the jth component of ith parallel subsystem

Numerical ex: R=0.9

then $R_{system} = [1-(1-0.9)^2]^4 = 0.96$

v.s. $R_{non\text{-}redundant} = (0.9)^4 = 0.6561$

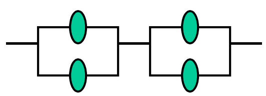
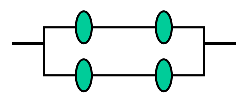
e.g.,



$$R_{parallel} = 1 - (1 - R_{series,\ 1})(1 - R_{series,\ 2})(1 - R_{series,\ 3})$$
$$= 1 - (1 - R_1^2)(1 - R_2^3)(1 - R_3)$$

Q: Prove the following theorem:

Replication at the component level is more effective than replication at the system level in improving system reliability using the same # of components.

Ex:  is better than 

Assume R=1/2 for each component

$$R_{system} = \left[1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\right]^2 = \frac{9}{16}$$

$$R_{system} = 1 - \left(1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right)^2 = \frac{7}{16}$$

D. k-out-of-n

e.g., TMR (Triple Module Redundancy) is a 2-out-of-3 system.

$$
\begin{aligned}
R_{TMR}(t) = \; & \underbrace{R_1(t) * R_2(t) * R_3(t)}_{\text{all are functioning}} \\
& \left.\begin{aligned}
&+ R_1(t)\,R_2(t)\,(\,1 - R_3(t)\,) \\
&+ R_1(t)\,R_3(t)\,(\,1 - R_2(t)\,) \\
&+ R_2(t)\,R_3(t)\,(\,1 - R_1(t)\,)
\end{aligned}\right\}\ \text{1 failed \& 2 are functioning}
\end{aligned}
$$

when $R_1(t) = R_2(t) = R_3(t) = R(t)$

$$R_{2\text{-out-of-}3}(t) = 3R^2(t) - 2R^3(t)$$

In general,

$$\boxed{R_{k\_out\_of\_n}(t) = \sum_{i=k}^{n} \binom{n}{i} R(t)^i (1 - R(t))^{n-i}} \quad \text{—— } 3$$

e.g.,

$$R_{2\_out\_of\_3} = \binom{3}{2} R^2(1 - R) + \binom{3}{3} R^3 = \frac{3!}{2!\,1!}(R^2 - R^3) + R^3 = 3R^2 - 2R^3$$

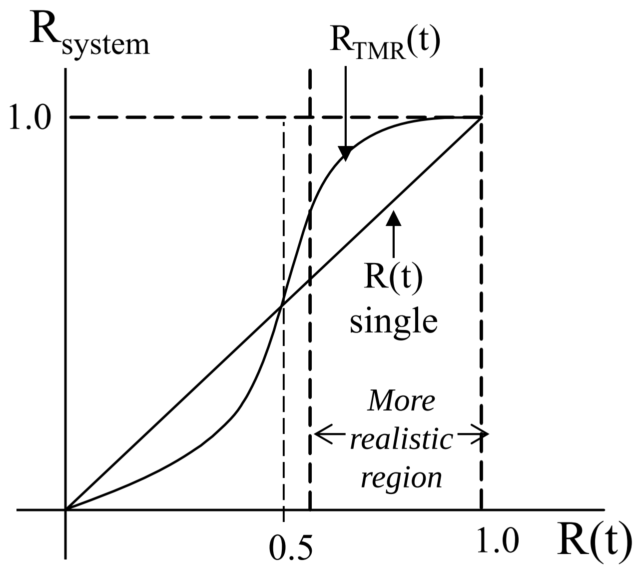
Q: Is $R_{TMR}(t) > R_{\text{system with a single component}}(t)$?

Let $3R^2 - 2R^3 = R$

$\therefore R^2 - \frac{3}{2}R + 0.5 = 0 \Rightarrow R = 1.0 \text{ or } 0.5$

$R_{TMR}(t) = R(t)$ when the reliability of a single module is 1.0 or 0.5

In fact, when $R(t) < 0.5$, $R(t) > R_{TMR}(t)$



Q: what is the MTTF of a k-out-of-n system when each single module follows the exponential failure law with a failure rate of λ ?

$$R_{system}(t) = \sum_{i=k}^{n} \frac{n!}{(n-i)!\,i!}(e^{-\lambda t})^i (1-e^{-\lambda t})^{n-i}$$

$$\therefore MTTF = \int_0^\infty R_{system}(t)dt = \frac{1}{\lambda}\left(\frac{1}{n}+\ldots+\frac{1}{k}\right)$$

Ex:

2-out-of-3:

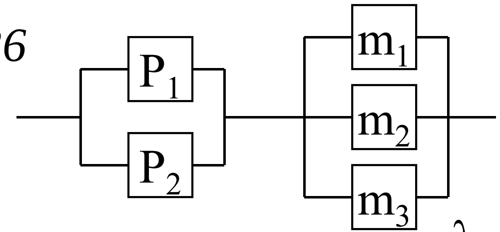
MTTF= $\frac{1}{\lambda}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5}{6}\frac{1}{\lambda}$

2-out-of-5:

MTTF= $\frac{1}{\lambda}\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}\right) = \frac{77}{60}\frac{1}{\lambda}$

Q: what is the reliability & MTTF of the following structure?

Fig. 2.5, p36



Also a parallel system

$\lambda_p = 0.00139$/day
(1-out-of-2)

A parallel system too

$\lambda_m = 0.00764$/day
(1-out-of-3)

$$R_{system}(t) = [\ 1 - (\ 1 - e^{-0.00139t}\ )^2\ ][\ 1 - (\ 1 - e^{-0.00764t}\ )^3\ ]$$

$$= 6e^{-0.00903t} - 3e^{-0.01042t} - 6e^{-0.01667t} + 3e^{-0.01806t} + 2e^{-0.02431t} - e^{-0.0257t}$$

$\Downarrow$

$$\text{MTTF} = \int_0^\infty R_{system}(t)dt$$

$$= \frac{6}{0.00903} - \frac{3}{0.01042} - \frac{6}{0.01667} + \frac{3}{0.01806} + \frac{2}{0.02431} - \frac{1}{0.0257} = 226.09$$

Recall that

Repair rate

Failure rate of component *i*

$$A_i(t) = \frac{\mu_i}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} e^{-(\lambda_i + \mu_i)t}$$

Equations 1 2 & 3 obtained above can also be used to compute the system availability $\rightarrow$ by replacing $R_i(t)$ with $A_i(t)$

Specifically,

$$A_{series}(t) = \prod_{i=1}^{n} A_i(t) \quad \text{④}$$

$$A_{parallel}(t) = 1 - \prod_{i=1}^{n} (1 - A_i(t)) \quad \text{⑤}$$

$$A_{k\_out\_of\_n}(t) = \sum_{i=k}^{n} \binom{n}{i} A^i(t)(1 - A(t))^{n-i} \quad \text{⑥}$$

Assuming $A_1(t) = A_2(t) = \ldots = A_n(t) = A(t)$

For steady state availability

Use $A_i(t \to \infty) = \dfrac{\mu_i}{\lambda_i + \mu_i}$ into equations ④ ⑤ & ⑥ above