Non-State-Space Models

1) Reliability block diagrams
2) Fault trees
3) Reliability graphs

Can be analyzed based on the individual components & info. about the system structure; the assumption is that the failure or repair of a component is not affected by other components.

State-Space Models

1) Markov — the "sojourn" time, i.e., the amount of time in a state, is exponentially distributed.
Chap. 4

2) Semi-Markov — the "sojourn" time, i.e., the amount of time in a state can be any distribution.
Chap. 8

When we associate "rewards" with states of Markov or Semi-Markov models, we have so called Markov reward models.
Chap. 6

3) Stochastic Petri Net Models — a concise & more intuitive representation for the Markov model.
Chap. 7

When we associate "rewards" to the markings of the net, we have stochastic reward nets.

Markov Models (continuous-time)

Two main concepts in the Markov model are "system state" and "state transition".

Representing the change of state due to the occurrence of an event, e.g., failures, repairs, etc.

Used to describe the system at any time. For reliability models, we frequently use faulty & non-faulty modules in the system.

Ex: TMR

System state representation:

$$(S_1, S_2, S_3) \text{ where } S_i = \begin{cases} 1 & \text{if module i is fault free} \\ 0 & \text{if module i is faulty} \end{cases}$$

States in which the system is operational	States in which the system has failed
(1,1,1)	(0,0,0)
(1,1,0)	(0,0,1)
(0,1,1)	(0,1,0)
(1,0,1)	(1,0,0)

How many of these?

2^n ← n is # of components in the state representation

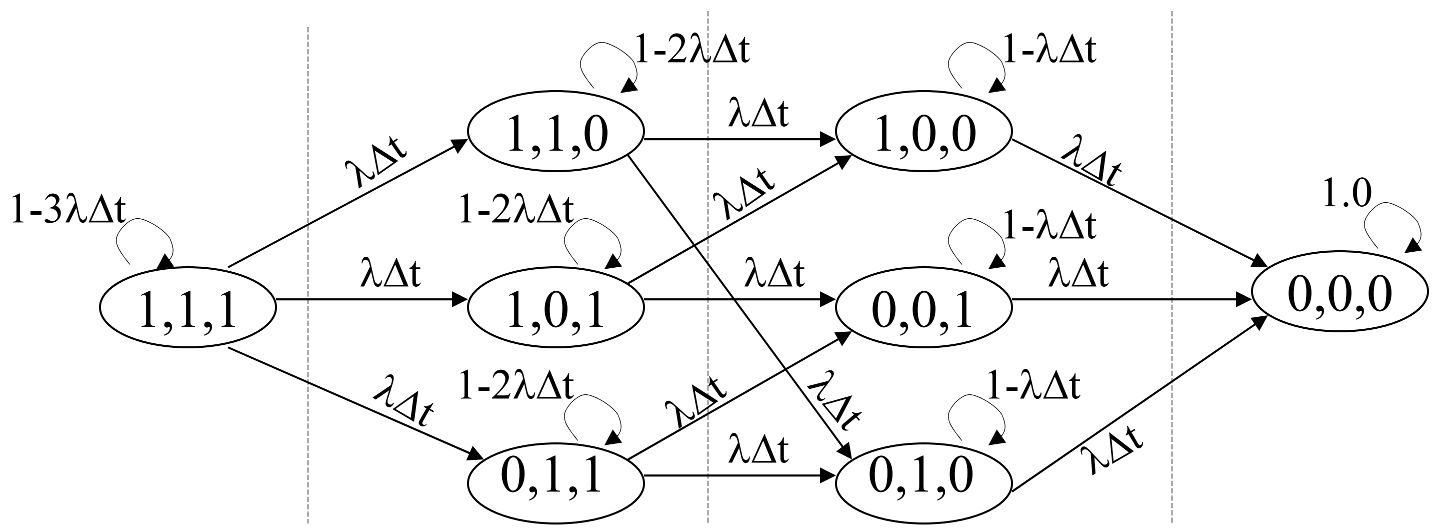
State transition

$$(1,1,1) \xrightarrow{\text{when module 1 fails}} (0,1,1)$$

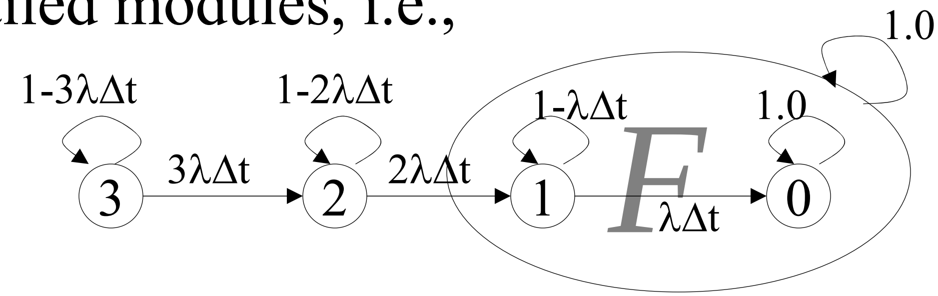
Assume that each module obeys the exponential failure law and has a constant failure rate λ. The prob. of module 1 being failed at time t+Δt, given that it was operational at time t, is given by $1 - e^{-\lambda \Delta t} \approx 1 - (1 + (-\lambda \Delta t) + \frac{(-\lambda \Delta t)^2}{2!} + \ldots) \approx \lambda \Delta t$

The prob. that a transition will occur is determined by the <u>prob. of failure</u>, <u>fault coverage</u>, <u>prob. of repair</u>, etc.

∴ Assume only one failure at a time. Then the state diagram of TMR is as follows:



The Markov model can be simplified by combining states having the same # of non-failed modules, i.e.,



The aggregate transition rate is from the perspective of source state; there is only a single component in the state representation.

$$\text{prob}\{\text{system in state j at } t+\Delta t\} = \sum_i \text{prob}\{\text{system was in state i at } t\} * \text{prob}\{\text{a single transition from i to j occurs within } \Delta t\}$$

e.g. $$\underbrace{P_3(t+\Delta t)}_{\substack{\text{System is in}\\ \text{state 3}\\ \text{at time } t+\Delta t}} = \underbrace{(1-3\lambda\Delta t)}_{\substack{\text{Prob. of}\\ 3\to 3 \text{ occurs}\\ \text{within time } \Delta t}} \underbrace{P_3(t)}_{\substack{\text{System was}\\ \text{at state 3}\\ \text{at time } t}}$$

$$P_2(t+\Delta t) = 3\lambda\Delta t P_3(t) + (1-2\lambda\Delta t)P_2(t)$$

$$P_F(t+\Delta t) = 2\lambda\Delta t P_2(t) + P_F(t)$$

Rewriting the above three expressions, we have:

$$\frac{dP_3(t)}{dt} = \lim_{\Delta t \to 0} \frac{P_3(t+\Delta t) - P_3(t)}{\Delta t} = -3\lambda P_3(t) \qquad (1)$$

$$\frac{dP_2(t)}{dt} = \lim_{\Delta t \to 0} \frac{P_2(t+\Delta t) - P_2(t)}{\Delta t} = 3\lambda P_3(t) - 2\lambda P_2(t) \qquad (2)$$

$$\frac{dP_3(t)}{dt} = \lim_{\Delta t \to 0} \frac{P_F(t+\Delta t) - P_F(t)}{\Delta t} = 2\lambda P_2(t) \qquad (3)$$

Or in matrix form as

$$\begin{bmatrix} P_3'(t) \\ P_2'(t) \\ P_F'(t) \end{bmatrix} = \begin{bmatrix} -3\lambda & 0 & 0 \\ 3\lambda & -2\lambda & 0 \\ 0 & 2\lambda & 0 \end{bmatrix} \begin{bmatrix} P_3(t) \\ P_2(t) \\ P_F(t) \end{bmatrix}$$

or $\boxed{\boldsymbol{P}'(t) = \boldsymbol{A}\boldsymbol{P}(t)}$

* this can be derived directly from the following <u>state-transition-rate</u> diagram

$$3 \xrightarrow{3\lambda} 2 \xrightarrow{2\lambda} F$$

<u>negative</u>: <u>out</u>
<u>positive</u>: <u>in</u>

$$\therefore P_3'(t) = -3\lambda P_3(t)$$
$$P_2'(t) = 3\lambda P_3(t) - 2\lambda P_2(t)$$
$$P_F'(t) = 2\lambda P_2(t)$$

The set of differential equations can be solved numerically or analytically. To solve it analytically, one approach is to use <u>Laplace Transform</u>.

$$F(t) \underset{\text{inverse LT}}{\overset{\text{LT}}{\rightleftarrows}} L(F(t)) = f(s)$$

Time domain			Laplace domain
	1	$\frac{1}{s}$	
	t	$\frac{1}{s^2}$	
	t^n	$\frac{n!}{s^{n+1}}$	
	e^{at}	$\frac{1}{s-a}$	

Laplace transform of derivatives:

if $L(F(t)) = f(s)$, then $L(F'(t)) = sf(s)-F(0)$

e.g., if $L(P_3(t))=P_3(s)$, then $L(P_3'(t)) = sP_3(s)-P_3(0)$

$\therefore$ Applying LT, we have

$$sP_3(s)-P_3(0) = -3\lambda P_3(s)$$
$$sP_2(s)-P_2(0) = 3\lambda P_3(s)-2\lambda P_2(s)$$
$$sP_F(s)-P_F(0) = 2\lambda P_F(s)$$

Where $P_3(s)$ is the LF of $P_3(t)$

$$\therefore\ P_3(s) = \frac{1}{s+3\lambda}$$

$$P_2(s) = \frac{3\lambda}{(s+2\lambda)(s+3\lambda)} = \frac{3}{s+2\lambda} + \frac{-3}{s+3\lambda}$$

$$\&\ P_F(s) = \frac{6\lambda^2}{s(s+2\lambda)(s+3\lambda)} = \frac{1}{s} + \frac{-3}{s+2\lambda} + \frac{2}{s+3\lambda}$$

Apply the inverse LT

$$\therefore\ P_3(t) = e^{-3\lambda t}$$

$$P_2(t) = 3e^{-2\lambda t} - 3e^{-3\lambda t}$$

$$P_F(t) = 1 - 3e^{-2\lambda t} + 2e^{-3\lambda t}$$

$$3 \xrightarrow{3\lambda} 2 \xrightarrow{2\lambda} F$$

For the TMR system, the system reliability is the sum of

$P_3(t) + P_2(t)$, i.e., $1 - P_F(t)$

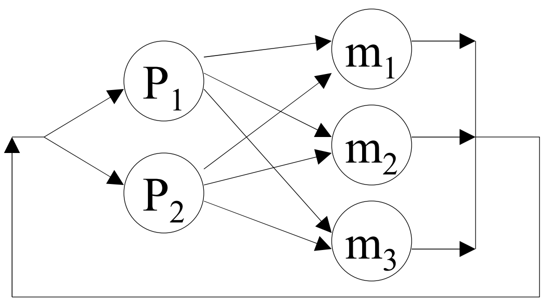
$$R_{system} = e^{-3\lambda t} + 3e^{-2\lambda t} - 3e^{-3\lambda t}$$
$$= \boxed{3e^{-2\lambda t} - 2e^{-3\lambda t}}$$

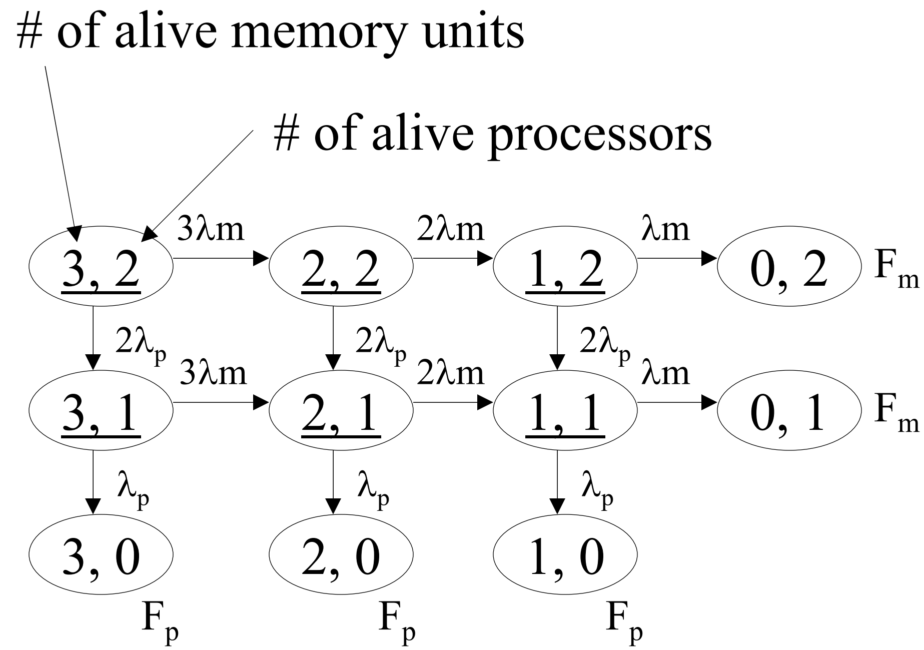
Same expression as we obtained earlier using a reliability block diagram or a fault tree model.

<u>In sharpe:</u>

```
markov   main(lambda)
   3    2    3*lambda
   2    F    2*lambda
end
   3    1.0
end
* print cdf=F(t) in symbolic form
  cdf(main;0.000001)                 * same as cdf(main,F;0.000001)
* print F(t) at t = 0.2, 0.4, 0.6, 0.8, 1.0
  eval(main,F;0.000001)  0.2   1.0   0.2
  end
```

Example: the 2P3m system





Modeling 1-out-of-3 memory & 1-out-of-2 CPU: the system is alive when at least one memory and one CPU are alive

$$R_{system}(t) = P_{32}(t) + P_{31}(t) + P_{22}(t) + P_{21}(t) + P_{12}(t) + P_{11}(t)$$

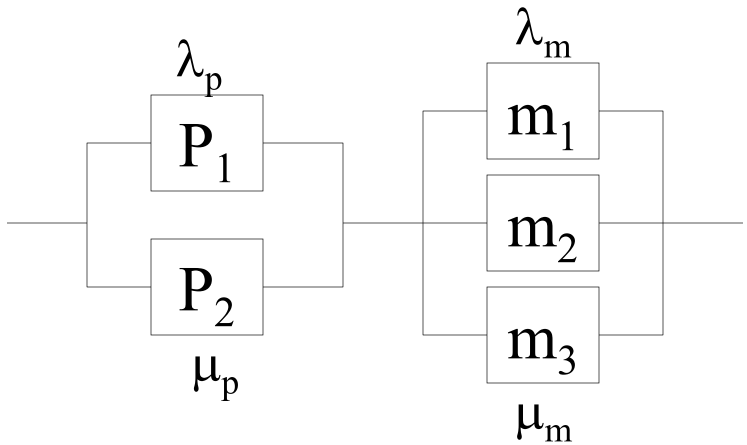
```
bind
lambdap     1/720           * MTTF of a processor
lambdam     1/(2*720)       * is 720 hrs
end                         * MTTF of a memory
                            * unit is 2*720 hrs

markov  2P3m
* memory failure
  32  22  3*lambdam
  22  12  2* lambdam
  12  02  lambdam
  31  21  3* lambdam
  21  11  2* lambdam
  11  01  lambdam
* processor failure
  32  31  2*lambdap
  31  30  lambdap
  22  21  2 * lambdap
  21  20  lambdap
  12  11  2 * lambdap
  11  10  lambdap
end
32  1.0
end
```

value(t; 2P3m) is the prob. of being in an absorbing state at time t;
value(t; 2P3m, 32) is the prob. of being in state 32 at time t.

```
* Q(t)
echo Q(t) is as follows:
cdf (2P3m)
* R(t) can be found by
* "expr 1-value(t;2P3m)";
* it can also be found by
* defining my own function
* called gp(t) below
func gp(t) value(t;2P3m,32)\
           +value(t;2P3m,22)\
           +.....\
           +value(t;2P3m,11)
* R(1 hr)
* print reliability(t=1 hr)
expr    1-value(1;2P3m)
* use loop to print R(t) at
* different values
loop    t, 0.5, 1, 0.1
expr    gp(t)
end
end
```

Availability Modeling



k-out-of-n memory subsystem

Case 1: Independent repairman model, i.e., all components have own repair facility and can be repaired independently

unavailability

$$U_p(t) = \frac{\lambda_p}{\lambda_p + \mu_p} - \frac{\lambda_p}{\lambda_p + \mu_p} e^{-(\lambda_p + \mu_p)t}$$

$$U_m(t) = \frac{\lambda_m}{\lambda_m + \mu_m} - \frac{\lambda_m}{\lambda_m + \mu_m} e^{-(\lambda_m + \mu_m)t}$$

```
bind
lambdap    1/720
lambdam    1/(2*720)
mup        1/4      ←MTTR = 4 hrs
mum        1/2
end
poly  U(lambda,mu)  gen\
       lambda/(lambda+mu), 0, 0\
     -lambda/(lambda+mu), 0, -(lambda+mu)
block      case1 (k,n)
comp       proc  U(lambdap, mup)
comp       mem U(lambdam, mum)
parallel   procs   proc   proc
kofn       mems    k, n, mem
series     top     procs   mems
end
loop       k, 1,3,1
* availability at the steady state (when t = ∞)
           expr pinf(case1; k, 3)
* instantaneous availability at t=100
           expr 1-value(100;case1;k, 3)
end
end
```

See p.354 text on a user-defined distribution <u>syntax</u>:

poly name(param-list) $\underbrace{\text{dist.}}$

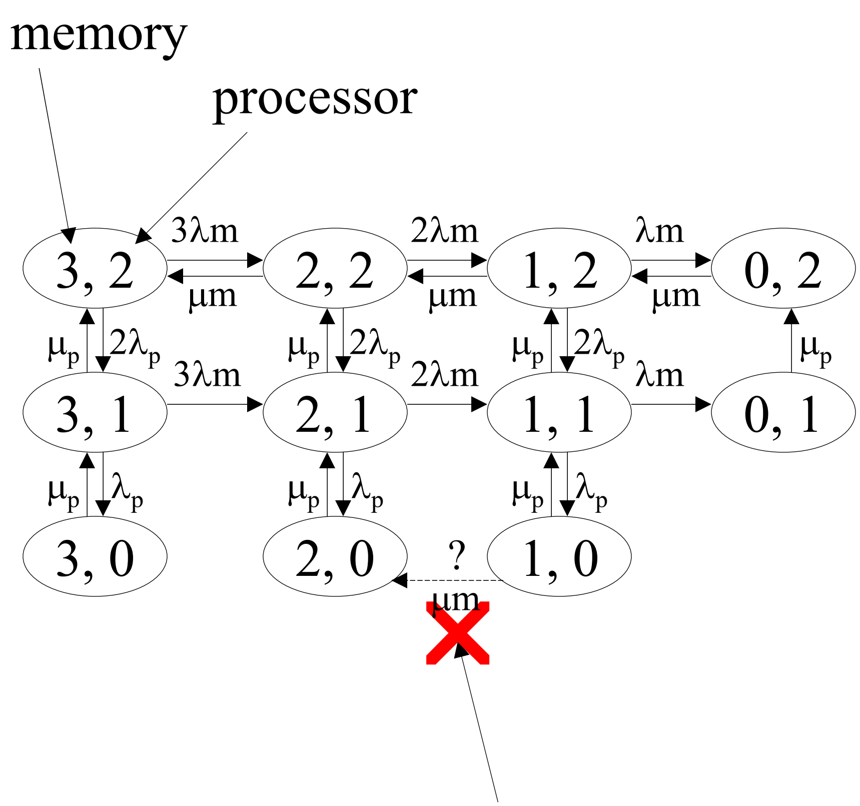
gen\
triple\
triple

of the form
a_j, k_j, b_j

$$F(t) = \sum_j a_j \cdot t^{k_j} \cdot e^{b_j t}$$

When defining a component, use unreliability F(t) for reliability modeling, and use unavailability U(t) or $\overline{A}(t)$ for availability modeling.

Case 2: There is only 1 repair facility capable of repairing one component at a time, with processor repair having a higher priority over memory repair.



No, because processor repair takes priority over memory repair

Assume that the system is up when at least 1 processor & 1 memory are up.

When the system is in a failure state, it halts until it is repaired to become operational again, so no further component failure will occur in a failure state

Same as before in the 2P3m Markov model for reliability modeling

```
bind
  lambdap     1/720
  lambdam     1/(2*720)
  mup         1/4
  mum         1/2
end
markov  M
* memory failure
32  22  3*lambdam
      ⋮
* processor failure
      ⋮
* processor repair
30  31  mup
31  32  mup
20  21  mup
21  22  mup
10  11  mup
11  12  mup
01  02  mup
* memory repair
22  32  mum
12  22  mum
02  12  mum
end
```

```
* steady state unavailability
  expr  prob(M,30)+prob(M,20)+\
        prob(M,10)+prob(M,01)+prob(M,02)
* for unavailability at time t = 1 hr
  expr  tvalue(1; M, 30)\
       +tvalue(1; M, 20)\
       +tvalue(1; M, 10)\
       +tvalue(1; M, 02)\
       +tvalue(1; M, 01)
  end
```

Sharpe code for availability modeling of Case 2

Modeling Near-Coincident Fault using a Markov Model

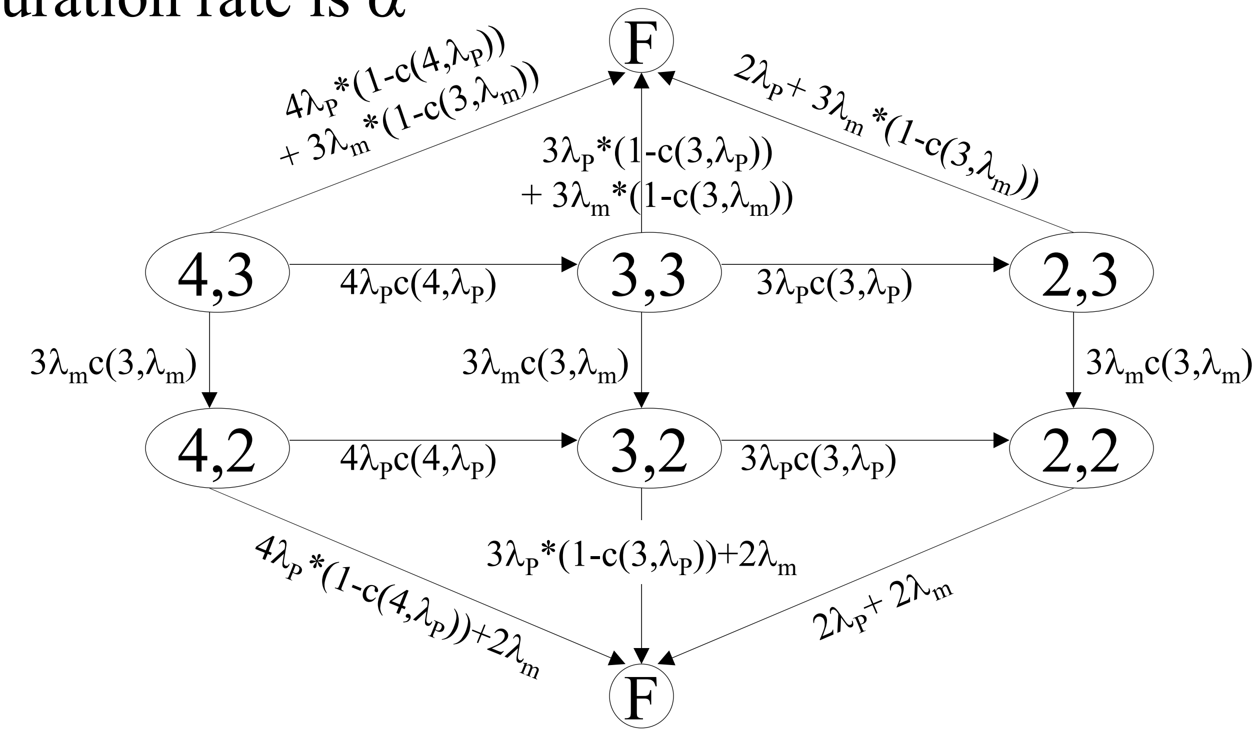
System Description: (Section 9.4.1)

1. 4 CPUs & 3 memories (λ_p & λ_m are failure rates). The system must have at least 2 CPUs & 2 memories working.

2. When a CPU or memory fails, the system can reconfigure to remove the failed component.

3. Reconfiguration fails iff a second failure of the same component type (as the failed component) occurs during the reconfiguration period. The system cannot cope with such a near-coincident fault, i.e., the system fails if such a near-coincident fault occurs during the reconfiguration period.

4. Reconfiguration rate is α



Here c(n,λ) means the <u>coverage factor</u> when 1 out of n components (with failure rate λ) fails: it is the probability that the system can successful perform a reconfiguration using the remaining n-1 components.

$$c(n, \lambda) = \frac{\alpha}{\alpha + (n-1)\lambda}$$

Time to occur:

$$
\begin{tikzcd}
\text{fault} \arrow[r, "\substack{(T_2)\\ \alpha}"] \arrow[d, "(n-1)\lambda", "(T_1)"'] & \text{recovered} \\
F &
\end{tikzcd}
$$

Probability of

$$\text{fault} \longrightarrow \text{recovered}$$

$$= \frac{\alpha}{(n-1)\lambda + \alpha}$$

$$P_{fault}(t) = e^{-((n-1)\lambda+\alpha)t}$$

$$P_{recovered}(t) = \frac{\alpha}{(n-1)\lambda+\alpha}(1-e^{-((n-1)\lambda+\alpha)t})$$

$$R_F(t) = \frac{(n-1)\lambda}{(n-1)\lambda+\alpha}(1-e^{-((n-1)\lambda+\alpha)t})$$

$\therefore$ when $t = \infty$

In general (even for non-exponential distribution)

prob of fault $\rightarrow$ recovered

$$= \text{prob}\ \{T_2 < T_1\}$$

$$= \int_0^\infty \text{prob}\ \{t < T_1\} \underbrace{f_{T_2}(t)}_{\textit{pdf}\text{ of } T_2} dt$$

$$= \int_0^\infty \underbrace{e^{-(n-1)\lambda t}}_{\text{prob. } \{t < T_1\}} \cdot \overbrace{\alpha e^{-\alpha t}}^{\textit{pdf}\text{ for } T_2} dt$$

$$= \frac{\alpha}{(n-1)\lambda + \alpha}$$

Laplace Tranform for F(t) is

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

$$\text{if} \quad F(t) = e^{-at}$$

$$\text{then} \quad f(s) = \int_0^\infty e^{-st} \cdot e^{-at}\,dt$$

$$= \frac{1}{s+a}$$

Sharpe code:

```
func  c(n, λ)\
  alpha/(alpha+(n-1)*λ)
bind
  alpha   360
end
markov   sift(λp,λm)
  43        33          4*λp*c(4,λp)
  33        23          3*λp*c(3,λp)
  42        32          4*λp*c(4,λp)
  32        22          3*λp*c(3,λp)
  43        42          3*λm*c(3,λm)
  33        32          3*λm*c(3,λm)
  23        22          3*λm*c(3,λm)
* to failure state
  43        F           4*λp*(1-c(4,λp))+3* λm*(1-c(3,λm))
  33        F           3*λp*(1-c(3,λp))+3* λm*(1-c(3,λm))
  23        F           2*λp+3* λm*(1-c(3,λm))
  42        F           4*λp*(1-c(4,λp))+2* λm
  32        F           3*λp*(1-c(3,λp))+2* λm
  22        F           2*λp+2* λm
end
  43       1.0
end
expr       mean(sift, F; 0.0001, 0.00001)
expr       1-value(10;sift;0.0001,0.00001)
end
```