Solution to Test #1 (open books/notes)

Part I: Multiple-Choice Problems (60%). Select one correct answer in each problem.

1. A system has three components, 1, 2, and 3. The system is available if component 2 is available and at least one of the two remaining components, 1 and 3, is available. Let $A_1$, $A_2$ and $A_3$ be the availabilities of components 1, 2 and 3, respectively. What is the availability of the system?
   (a) $A_1 A_3 + A_2 A_3 - A_1 A_2 A_3$
   (b) $A_1 A_2 + A_2 A_3 - A_1 A_2 A_3$
   (c) $A_2 (1 - A_1)(1 - A_3)$
   (d) $A_2 (1 - A_1 A_3)$
   (e) $1 - (1 - A_1)(1 - A_2)(1 - A_3)$

2. Use M/M/1 to solve this problem. Consider a phone booth servicing customers arriving at a rate of 20 customers/hour. Suppose the average duration of a phone conversation is 2 minutes. What is the average wait time (just the queuing time) per customer at the phone booth?
   (a) 1 min
   (b) 2 min
   (c) 3 min
   (d) 4 min
   (e) 5 min

3. Suppose we observe a system for 1000 minutes. Within this period, the system is busy for 800 minutes, the total number of customers completed is 5000 and the average customer population is 10. Which of the following is not true?
   (a) the utilization of the system is 0.8
   (b) the throughput of the system is 5 customers/minute
   (c) the response time per customer is 2 minute
   (d) the wait time per customer is 0.2 minute

4. A server system obeying exponential failure law has a constant failure rate of $0.1 \text{ hr}^{-1}$. Assume that customers arrive at the server system with a rate of 10 customers/hour and the server is able to service customers at a rate of 12 customers/hour. Which one of the following is true?
   (a) The reliability of the system $1 - e^{-0.1}$ at time $= 1$ hour
   (b) The expected number of customers the system can serve before it fails is 100 customers
   (c) The average population in the system before it fails is 10 customers
   (d) the expected response time per customer before it fails is 1 hour.
   (e) The throughput of the system before it fails is 12 customers/hour

5. For an M/M/3/8 queueing facility with arrival rate of $\lambda$ and service rate of $\mu$ for each server, which one of the following is false? Assume that $P_i$ stands for the steady-state probability that the system contains $i$ customers (including both queueing and in service).
(a) the average number of customers is $\sum_{i=0}^{8} (i \times P_i)$

(b) the average rejection rate is $P_8 \times \lambda$

(c) the throughput is $\sum_{i=1}^{3} (i \times \mu \times P_i) + \sum_{i=4}^{8} (3\mu \times P_i)$

(d) the throughput is $\sum_{i=0}^{7} (\lambda \times P_i)$

(e) the utilization of the facility is $\sum_{i=0}^{8} P_i$

6. Which of the following statements is false for the fault tree model shown above?
   (a) the system is alive when components 1, 3, and 4 are alive
   (b) the system fails when component 4 and 5 fail
   (c) the system is alive when components 2, 3, and 5 are alive
   (d) the system fails when component 3 fails
   (e) the system fails when component 4 fails

7. Continue from the above question. Suppose that components 1, 2, and 3 are always reliable with probability 1 (say via internal redundancy) while components 4 and 5 have the reliabilities of 0.8 and 0.7, respectively, at 2000 hours. What is the system reliability at 2000 hours?
   (a) 0.006
   (b) 0.994
   (c) 0.504
8. Consider a TMR system with each component having an independent failure rate of $\lambda$ and an independent repair rate of $\mu$. Assume that alive components can still fail regardless of the states of other components. A Markov model for this TMR system is shown below to model the availability of the system. The system is operational when at least 2 out of 3 components are operational. State $i$ means that $i$ components are functional.

Let $P_i(t)$ be the probability that the system is in state $i$ at time $t$ based on the above Markov model. What is the unavailability of the system at time $t$?

(a) $e^{-3\lambda t}$
(b) $P_2(t) + P_3(t)$
(c) $1 - e^{-3\lambda t} + e^{-\mu t} - e^{-2\lambda t}$
(d) $P_0(t) + P_1(t)$
(e) cannot be calculated from the Markov model.

9. Continue from the same Markov model. What is the unreliability of the system at time $t$?

(a) $e^{-3\lambda t}$
(b) $P_2(t) + P_3(t)$
(c) $1 - e^{-3\lambda t} + e^{-\mu t} - e^{-2\lambda t}$
(d) $P_0(t) + P_1(t)$
(e) cannot be calculated from the Markov model.

10. Continue from the same Markov model. Suppose we assign a reward of zero to states 1 and 0, and a reward of one to states 3 and 2. The expected cumulative reward $E[Y(t)]$ under this reward assignment is equal to

(a) the mean down time in $[0,t]$
(b) the mean up time in $[0,t]$
(c) the mean number of jobs which the system can service during $[0,t]$
(d) the mean time to failure
(e) the availability of the system at time $t$

11. Consider an open queueing network system shown below that has a product-form solution.

As shown in the figure, the external arrival rate to the system is 6 jobs/sec and the service rates at the two centers are 20 jobs/sec and 8 jobs/sec, respectively. When a job departs center 1, it completes its service (and thus leaves the system) with probability 0.6 or conversely goes to center 2 with probability 0.4. When a job departs center 2, it goes to center 1 with probability 1. Which of the following is false for the system described above?

(a) throughput at center 1 is 10 jobs/sec
(b) population at center 1 is 1
(c) response time at center 1 per visit is 1/10 sec
(d) response time at center 2 per visit is 1/4 sec
√ (e) response time is 1/2 sec
12. Consider the closed QNM below consisting of a terminal center (T) with M terminals and a central subsystem (F, C, D and P). Suppose that the visit counts to centers T, F, C, D, P by a client are \( v_T, v_F, v_C, v_D, \) and \( v_P \), respectively, with \( v_T = 1 \). The average response times per job per visit at centers T, F, C, D, P are \( r_T, r_F, r_C, r_D, \) and \( r_P \), respectively. Which one of the following is not true?

(a) the throughput of the terminal center is \( \frac{M}{v_T r_T + v_F r_F + v_C r_C + v_D r_D + v_P r_P} \)

(b) the throughput of center D is \( v_D \times \frac{M}{v_T r_T + v_F r_F + v_C r_C + v_D r_D + v_P r_P} \)

(c) the average amount of time a terminal job stays in the central subsystem is \( v_F r_F + v_C r_C + v_D r_D + v_P r_P \)

(d) the throughput of the central subsystem is \( v_F p_0 \times \frac{M}{v_T r_T + v_F r_F + v_C r_C + v_D r_D + v_P r_P} \)

√ (e) \( v_C = P_0 v_T + P_1 v_F + v_D + v_P \)
1. (15 points.) Consider a system with 8 components connected in a structure as shown in the above block diagram. Assume that component \( i \) has an independent failure rate \( \lambda_i \) and an independent repair rate \( \mu_i \).

(a) Identify the minimal path set for this system.

(b) Draw a fault tree model based on the minimal path set for calculating the availability of the system. Label your fault tree nodes and component names properly.

Ans: (a) the minimal path set is \( \{1,3,5,8\} \), \( \{1,3,6,7\} \), \( \{1,4,6,5,8\} \), \( \{1,4,7\} \), \( \{2,3,4,7\} \), \( \{2,5,8\} \), \( \{2,6,7\} \).

(b) a diagram is not shown here but the fault tree is an AND gate on top with 7 OR gates at the bottom each covering a minimal path above.
2. (10 points.) Consider a special M/M/2/3 queueing facility. There are two servers in the facility that serve customers in tandem, that is, when a job completes at server 1 then it goes to server 2. After a job completes at server 2, it departs. The system can at most hold three jobs at any time. The job arrival rate is $\lambda$ and the service rates of the two servers are $\mu_1$ and $\mu_2$, respectively, for server 1 and server 2.

(a) Draw a Markov model, using $(i, j)$ as the state representation with $i$ denoting the number of jobs at server 1 (and the associated queue) and $j$ denoting the number of jobs at server 2.

(b) Let $P_{(i,j)}$ be the steady-state probability that there are $i$ jobs at server 1 (and the associated queue) and $j$ jobs at server 2. Give mathematical expressions for:

i. the rejection probability
ii. the response time

**Ans: (a)**

![Markov Model Diagram](image)

**Ans: (b)**

Rejection probability $= P_{(3,0)} + P_{(2,1)} + P_{(1,2)} + P_{(0,3)}$

Throughput $= \mu_2 \times \left( P_{(0,3)} + P_{(1,2)} + P_{(2,1)} + P_{(1,1)} + P_{(0,1)} \right)$

Population $= 3 \times \left( P_{(0,3)} + P_{(1,2)} + P_{(2,1)} + P_{(3,0)} \right) + 2 \times \left( P_{(0,2)} + P_{(1,1)} + P_{(2,0)} \right) + 1 \times \left( P_{(0,1)} + P_{(1,0)} \right)$

Response time $= \text{Population}/\text{Throughput}$.
3. (15 points.) Suppose that a network switch center has $n = 3$ slots to accommodate incoming high and low-priority clients, with arrival rates of $\lambda_h$ and $\lambda_l$ and departure rates of $\mu_h$ and $\mu_l$, respectively. All clients will each occupy one full slot, regardless of class. Draw a Markov state transition diagram for modeling the following resource control policy:

- If there is at least one slot available, an incoming client always occupies an empty slot, regardless of its priority class.
- If case all slots are filled, an incoming high-priority client can preempt a low-priority client if it is available, after which the high-priority client occupies one slot and the preempted low-priority client is forced to abort prematurely. If no low-priority client is available, the incoming high-priority client will be rejected.
- If case all slots are filled, an incoming low-priority client will be rejected.

Use the representation $(a, b)$ where $a$ stands for the number of low-priority clients, and $b$ stands for the number of high-priority clients. Organize the Markov model so that when a high priority client arrives, the transition goes right; and when a low priority client arrives, the transition goes down. Label the transition rate of each transition clearly.

Suppose $P_{(a,b)}$ is the steady state probability that the system is in state $(a, b)$. Give mathematical expressions using $P_{(a,b)}$ for calculating the following performance metrics:

- rejection probability of low-priority clients;
- abortion rate of low-priority clients;
- throughput of high-priority clients.
The rejection probability of low-priority clients is given by:

\[ P_{(3,0)} + P_{(2,1)} + P_{(1,2)} + P_{(0,3)} \]

The abortion rate of low-priority clients is given by:

\[ \lambda_h P_{(3,0)} + \lambda_h P_{(2,1)} + \lambda_h P_{(1,2)} \]

The throughput of high-priority clients is given by:

\[ \mu_h P_{(0,1)} + 2\mu_h P_{(0,2)} + 3\mu_h P_{(0,3)} + \mu_h P_{(1,1)} + 2\mu_h P_{(1,2)} + \mu_h P_{(2,1)} \]