Bundling in Expert Mediated Search

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Abstract. In search-based markets with noisy signals, like the market for used cars, experts can play an important role. These experts act as information brokers, revealing the true value of a good in exchange for the payment of a fee. Sometimes these experts may choose to sell only bundles of their services (three car inspections for a fixed price, e.g.). We analyze bundling of services in a model of expert-mediated (one-sided) search, and derive optimal strategies for experts and buyers. Our analysis reveals some surprising results. In particular, there are situations where offering only non-unit-size bundles of services can be pareto-improving for the expert and the buyer. Further, in markets with low search costs, the optimal strategy for the expert may well be to offer unlimited services for a single flat fee.

Keywords: One-sided search, Product/service bundling, Agent-mediated search

1 Introduction

Autonomous agents, acting on behalf of their users are commonly required to engage in costly search [5, 8, 13]. The search process is characterized by the need to sequentially evaluate opportunities in order to select one of them. The process of evaluating an opportunity commonly incurs a cost [1], and the searcher’s goal is to maximize the value of the opportunity eventually exploited minus the costs accumulated along the search process. The searcher thus trades off, on each step of its search, the possible marginal gain from revealing the values of further opportunities and the cost associated with doing so [15, 11, 14].

Consider a consumer looking to buy a used car. Typically she will visit potential sellers (or car dealers) in order to estimate the value of the car/s they offer. Visiting sellers and looking at cars incurs a cost (accumulated along the search process), either monetary or in terms of resources that need to be consumed. The search continues until the searcher eventually buys one of the cars seen. While traditional models of sequential search in costly settings assume that the searcher obtains the true value of an opportunity, in many realistic settings she only sees a noisy signal. For example, a non-expert consumer cannot evaluate the mechanical condition of a car. This uncertainty presents an opportunity for
the emergence of knowledge brokers or experts in both traditional and Internet marketplaces. In exchange for the payment of a fee, an expert can disambiguate the noisy signal, providing a better estimate of the true value than the noisy signal available to the non-expert consumer. For example, Carfax.com is a service offering car buyers the history records of any used car, thus enabling a better assessment of its true worth, for a fee. The existence of an expert can theoretically lead to substantially better overall outcomes for buyers [4].

This paper considers what happens when an expert can bundle its services. Bundling of goods and services is a common practice in the real world, ranging from daily necessities at supermarkets to digital information goods and services available online. There is a vast literature on bundling and its advantages and disadvantages from many perspectives [7, 2, 3]. However, research has typically focused on the bundling of complementary and substitute goods. Many studies have looked at profitability and discounts offered in bundling services that use common infrastructure, like phone, cable and internet service, or complementary services like flights, hotels and rental cars.

Bundling has a somewhat different meaning in the context of expert-mediated search. In such markets bundling means that, for a predefined fee, the searcher can use the expert services some fixed number of times. Such schemes are common in physical and virtual markets. For example, Carfax offers a bundle of five reports for $44.99 (alongside the option to buy a single report for $34.99). The bundle-based model presented and analyzed in this paper generalizes both the classic sequential search model [11, 10] and its extension to noisy environments.

The problem can be formulated as a Stackelberg game, where the expert makes the first move by setting the bundle size and price and the searcher responds. We characterize the searcher’s optimal search strategy, and use that to find the optimal strategy for the expert in terms of bundle size and price. It is not surprising that bundling often leads to an increase in the expected profit of a monopolist expert, but we also find that bundling can be pareto-improving, simultaneously increasing the expected utility of the buyer. While the expert sells bundles at higher prices, the buyer benefits because the extra amount she pays is more than made up for by the higher value of the opportunity eventually picked. We also find that, especially when search costs are low, the profit maximizing strategy for the expert can be to offer unlimited services for a fixed price.

2 The Model

The classic sequential model of one-sided search [15, 11, 14] considers a searcher facing an infinite stream of opportunities from which she needs to choose one. The true values of the different opportunities are initially unknown to the searcher. The searcher is acquainted, however, with the probability density function from which the true values are drawn, denoted \( f_v(x) \). The searcher can reveal the true value of an opportunity for a cost \( c_s \). Having no a priori information about any specific opportunity, the searcher reviews the opportunities she encounters sequentially. The problem of the searcher is thus to find a strategy \( S \) that maps the best value found so far to the decision \( \{ \text{terminate, resume} \} \), given \( c_s \) and
in a way that the expected value of opportunity eventually obtained minus the accumulated costs along the search is maximized.

The assumption that the value received is the true value of the opportunity encountered is sometimes relaxed by considering the value obtained to be a noisy signal \( s \), correlated to the true value by a known probability distribution function \( f_s(y|v) \) [4, 6, 9]. In these model variants, the searcher usually can obtain, for some additional cost, denoted \( c_e \) (e.g., using the services of an expert [4], or an interviewing [9] or dating [6] process), the true value (or at least a better estimate) of the opportunity for which signal \( s \) is received. In the expert case, which we consider here, we assume the expert can produce its valuation at a cost \( d_e \geq 0 \), and that the value revealed by the expert is the true value of the opportunity, rather than just a less noisy signal. Here, again, the goal of the searcher is to find a strategy that maps the best signal received so far to the decision \{terminate, resume, query\}, where “terminate” means stopping the search, “resume” means evaluating an additional opportunity (without the help of the expert) and “query” means asking the expert to reveal the true value.

We consider the case with no recall: if the searcher rejects one opportunity, she cannot go back to it later.\(^1\) Denoting the expected number of times the expert’s services are used by \( \eta_b \), the expert’s expected profit, \( \pi \), is given by:

\[
\pi = \eta_b (c_e - d_e).
\]

The goal of the expert is to find \( c^*_e \) that maximizes \( \pi(c^*_e) \).

We generalize the model above by introducing the option that experts may offer their services in packages (“bundles”). For a cost \( c^b_e \), the searcher obtains, upon purchasing the bundle, the right to use the expert’s service \( k \) times along its search path with no additional cost. The searcher does not necessarily need to make use of all the \( k \) queries, and similarly, if she uses them all she can purchase additional bundles as required. The goal of the searcher is to find an optimal strategy as before (mapping from a signal received to \{terminate, resume, query\}). When choosing to query the expert the agent needs to pay a cost \( c^b_e \) if she has not purchased a bundle yet or if she has already used the expert’s services \( k \) times since the last bundle was purchased. When considering the expert’s profit, we need to distinguish between the expected number of bundles purchased by the searcher, denoted \( \eta_b \), and the expected number of queries used by the searcher, denoted \( \eta_q \) (\( \eta_q \leq \eta_b k \)). The expert’s expected profit is now given by:

\[
\pi = c^b_e \eta_b - \eta_q d_e.
\]

The goal of the expert is to find the optimal pair \((k, c^b_e)\) that maximizes its overall expected profit \( \pi \).

### 3 Analysis and Optimal Policies

We begin by characterizing the optimal strategy of the searcher. Based on the analysis we prove that for a very reasonable assumption on signal structure, namely that “higher signals are good news” as formalized below, the optimal strategy can be represented by a set of reservation values. We then turn to analyzing the expert’s revenue given the bundle characteristics she sets.

\(^1\) This is often the case in real-life settings, e.g., a used car or an apartment seen cannot be guarantee to remain in the market for long.
3.1 Optimal Search Strategy

Since the searcher cannot recall previous opportunities, her state depends only on the number of remaining pre-paid queries, denoted $\gamma$. The system can thus be modeled as a Markov Decision Process with $k$ search states ($\gamma = 0, 1, 2, \ldots, k-1$) and one termination state, as illustrated in Figure 1. Upon receiving a signal $s$ when in state $\gamma > 0$, the searcher can either: (a) reject the current opportunity and continue search, starting from the same state $\gamma$; (b) accept the current opportunity and terminate search; (c) query the expert for the true value of the current opportunity and, based on the value received, either accept the current opportunity (terminating the search) or reject the current opportunity and continue search from state $\gamma - 1$. When in state $\gamma = 0$, the searcher has the same options when receiving a signal $s$, except that querying the expert incurs a cost $c_k$ and if the searcher chooses to resume search, based on the value received, she continues from state $\gamma = k - 1$.

![MDP representation of the searcher’s problem](image)

**Fig. 1.** MDP representation of the searcher’s problem (state variable is the number of remaining queries that can be used).

Let $V_\gamma$ denote the expected value of following the optimal search strategy starting from state $\gamma$ (the value-to-go). For simplicity, we work with the unconditional distribution of signals received, $f_s(x)$, and the distribution of true values conditional on the signals received), $f_v(y|x)$. First, observe that when the searcher is in state $\gamma > 0$, the only two applicable alternatives are querying the expert and resuming the search without querying the expert.

**Proposition 1.** When in state $\gamma > 0$, querying the expert dominates terminating search without querying the expert.

**Proof.** For any strategy that terminates search upon obtaining a signal $s$ when in state $\gamma > 0$, consider instead a modification of that strategy which queries the expert, and terminates only if the true value, revealed by the expert, is greater than $V_{\gamma-1}$. This new strategy clearly dominates, since if the true value is less than $V_{\gamma-1}$ the searcher is better off resuming search, and she pays no marginal cost to obtain the expert’s service in this instance. $\square$

For any signal $s$ received in state $\gamma > 0$, the expected benefit if querying the expert, denoted by $M(s, V_{\gamma-1})$, is given by: $M(s, V_{\gamma-1}) = \int_{-\infty}^{\infty} \max(x, V_{\gamma-1}) f_v(x|s) dx$.  

$^2$ These are interchangeable with the prior distribution of values, $f_v(y)$, and the distribution of signals conditional on values, $f_s(x|v)$, by Bayes’ Law.
Therefore, the optimal strategy is to query the expert if $M(s, V_{\gamma-1}) > V_\gamma$. The expected benefit of using the optimal strategy, when starting from state $\gamma > 0$, is thus given by:

$$V_{\gamma>0} = -c_s + V_\gamma \int_{M(s, V_{\gamma-1})<V_\gamma} f_s(s) \, ds + \int_{M(s, V_{\gamma-1})>V_\gamma} f_s(s) M(s, V_{\gamma-1}) \, ds$$  \hspace{1cm} (1)

Similarly, when in state $\gamma = 0$, the expected benefit of obtaining a signal $s$ is: (a) $E[x|s]$ if terminating the search without querying; (b) $V_0$ if resuming the search without querying the expert; and (c) $M(s, V_{k-1}) - c_e^k$ on querying the expert. For any signal $s$, the choice which yields the maximum among the three should be made. Let $\text{MAX} = \max(E[x|s], V_0, M(s, V_{k-1}) - c_e^k)$. Let $\zeta_1, \zeta_2$ and $\zeta_3$ define the sets of signal support such that $\text{MAX}$ equals $V_0, E[x|s]$, and $M(s, V_{k-1}) - c_e^k$, respectively. The expected benefit is then:

$$V_0 = -c_s + V_0 \int_{\zeta_1} f_s(s) \, ds + \int_{\zeta_2} E[x|s] f_s(s) \, ds + \int_{\zeta_3} f_s(s) (M(s, V_{k-1}) - c_e^k) \, ds$$  \hspace{1cm} (2)

The optimal strategy can be obtained by solving the set of $k - 1$ equation instances of Equation 1 (for $1 \leq \gamma < k$) in addition to Equation 2. An important property of the optimal strategy is given in the following Lemma.

**Lemma 1.** When using the optimal strategy, $V_{\gamma} \leq V_{\gamma+1} \forall \ 0 \leq \gamma < k - 1$.

**Proof.** Suppose $V_{\gamma} > V_{\gamma+1}$. A searcher starting from state $\gamma + 1$ can follow the optimal strategy as if starting from $\gamma$. In this case it will end up with the same value $v$ while the accumulated cost will be at worst equal and possibly less (in the case where search terminates after the last bundle purchase) than the accumulated cost when starting the search from state $\gamma$; therefore the original strategy could not have been optimal. $\square$

### 3.2 The HSGN case

While the structure of the optimal strategy tightly depends on $f_s(x|v)$, for some cases, the optimal strategy can have a simple representation in the form of reservation values. For example, suppose the standard assumption that “higher signals are ‘good news’” (HSGN) holds.\(^3\) This means that for any $s_1 > s_2$, the conditional distribution of $v$ given $s_1$ first-order stochastically dominates that of $v$ given $s_2$, i.e., $\forall y \ F_v(y|s_1) < F_v(y|s_2), \forall y$.

**Theorem 1.** For $f_v(y|s)$ satisfying the HSGN assumption, for any signal $s$, the optimal strategy for the searcher in state $\gamma$ can be described as (see Figure 2): (1) a tuple $(t_1, t_u, V_{k-1})$, corresponding to state $\gamma = 0$, such that for any signal obtained: (a) the search should resume if $s < t_1$; (b) the opportunity should be accepted if $s > t_u$; and (c) the expert should be queried if $t_1 < s < t_u$ and

\(^3\) This innocuous assumption loosely means that a higher signal implies a higher probability of a higher true value and is widely used in the literature [12, 16, 4].
Fig. 2. Characterization of the optimal strategy for noisy search with an expert offering bundles. For state 0 (left figure), the searcher queries the expert if $s \in [t_l, t_u]$ and terminates search if the worth is greater than the value of resuming the search $V_{k-1}$. The searcher resumes search if $s < t_l$ and terminates without querying the expert if $s > t_u$. However, for any state $\gamma > 0$ (right figure), the searcher rejects the opportunity (terminating search) if $s < t_{\gamma}$ and otherwise queries the expert before deciding.

the opportunity accepted (and search terminated) if the value obtained from the expert is above the expected value of resuming the search, $V_{k-1}$, otherwise search should resume; and

(2) a set of $(k-1)$ tuples $(t_\gamma, V_{\gamma-1})$ corresponding to states $\gamma \in 1, 2, \ldots, k-1$ such that: (a) the search should resume if $s < t_{\gamma}$; and (b) the expert should be queried if $s > t_{\gamma}$ and the opportunity accepted if the value obtained from the expert is above the expected value of resuming the search, $V_{\gamma-1}$, otherwise search should resume.

Proof. (a) The proof augments the one given in [4] regarding the strategy structure in cases where queries are sold only one at a time. We first show that if it is optimal for the searcher to resume search given a signal $s$, then it must also be optimal for her to do so given any other signal $s' < s$. Then, we show that if it is optimal for the searcher to terminate search given a signal $s$, then it must also necessarily be optimal for her to do so given any other signal $s'' > s$.

If the optimal strategy given signal $s$ is to resume search then $V_0 > \max(E[x|s], M(s, V_{k-1}) - c_k^e)$. From the HSGN assumption, $E[x|s] > E[x|s']$ (since $s' < s$). Similarly:

$$M(s, V_{k-1}) = \int_{x=V_{k-1}}^{\infty} x f_v(x|s) \, dx + V_{k-1} \int_{x=-\infty}^{V_{k-1}} f_v(x|s) \, dx$$

$$= V_{k-1} + \int_{x=V_{k-1}}^{\infty} (x - V_{k-1}) f_v(x|s) \, dx$$

$$> V_{k-1} + \int_{x=V_{k-1}}^{\infty} (x - V_{k-1}) f_v(x|s') \, dx$$

$$= \int_{x=V_{k-1}}^{\infty} x f_v(x|s') \, dx + V_{k-1} \int_{x=-\infty}^{V_{k-1}} f_v(x|s') \, dx$$

$$= \int \max(x, V_{k-1}) f_v(x|s') \, dx = M(s', V_{k-1})$$

The proof for $s'' > s$ is similar: the expected cost of accepting the current opportunity can be shown to dominate both resuming the search and querying the expert. We omit the details because of space considerations. The optimal strategy can thus be described by the tuple $(t_l, t_u, V_{k-1})$ as stated in the theorem.
and substituting \( V(a) \), ignoring the option to terminate the search without querying the expert, then she must also do so given any other signal \( s \). The proof is similar to (a), ignoring the option to terminate the search without querying the expert, and substituting \( V_{k-1} \) with \( V_{\gamma-1} \) whenever applicable. \( \square \)

Based on Theorem 1, we can construct the appropriate modifications of Equations 1 and 2 for the HSGN case:

\[
V_0 = -c_s + V_0 F_s(t_t) \, ds + \int_{s=t_u}^{\infty} f_s(s) E[v|s] \, ds - c_e \int_{s=t_t}^{t_u} f_s(s) \, ds + \int_{s=t_t}^{t_u} f_s(s) F_v(V_{k-1}|s) \, ds + \int_{s=t_t}^{t_u} f_s(s) \int_{x=V_{k-1}}^{\infty} x f_v(x|s) \, dx \, ds
\]

\[V_{k-1} = \int_{s=t_t}^{t_u} f_s(s) F_v(V_{k-1}|s) \, ds + \int_{s=t_t}^{t_u} f_s(s) \int_{x=V_{k-1}}^{\infty} x f_v(x|s) \, dx \, ds \]

\[
V_{\gamma>0} = -c_s + V_{\gamma} \int_{s=0}^{t_u} f_s(s) \, ds + \int_{s=0}^{t_u} f_s(s) \int_{x=V_{\gamma-1}}^{\infty} x f_v(x|s) \, dx \, ds + \int_{s=0}^{t_u} f_s(s) V_{\gamma-1} F_v(V_{\gamma-1}|s) \, ds
\]

where \( F_v(x|s) \) and \( F_s(s) \) are the appropriate c.d.f of \( f_v(x|s) \) and \( f_s(s) \), respectively.

The values \((t_t, t_u)\) and \( t_s, \forall \gamma > 0 \) to be used in the optimal strategy are obtained by deriving Equation 3 w.r.t \( t_t \) and \( t_u \) (separately) and Equation 4 w.r.t \( t_s, \forall \gamma > 0 \), resulting in (after integration by parts):

\[
c_e^k = \int_{y=V_{k-1}}^{\infty} (x-V_{k-1}) f_v(x|t_t) \, dx
\]

\[
c_e^k = \int_{y=V_{k-1}}^{-\infty} (V_{k-1}-x) f_v(x|t_u) \, dx
\]

\[
V_{\gamma} = V_{\gamma-1} F_v(\gamma-1|t_t) + \int_{V_{\gamma-1}}^{\infty} x f_v(x|t_u) \, dx
\]

Equations 5-7 can intuitively be interpreted as describing the indifference values of the searcher. Equation 5 characterizes the intuition that, at \( s = t_t \), the searcher is indifferent between either resuming the search or querying the expert. From equation 6 we see that at \( s = t_u \), the searcher is indifferent between either terminating the search or querying the expert. Finally, \( s = t_s \) in Equation 7 is the signal where the expected gain from rejecting the opportunity with which it is associated is equal to the expected gain received by deciding after querying the expert.

Based on Equations 3-7 we can construct a set of \( 2k+1 \) equations from which the optimal search strategy can be extracted. These are Equations 3,5-6 for state \( \gamma = 0 \) and the \( 2k-2 \) equations resulting from Equations 4 and 7 for \( \gamma = 1,2 \cdots k-1 \). We can solve the above system of equations to calculate the equilibrium of the Stackelberg game. The searcher’s utility in this case is given by \( V_0 \) because the searcher starts from state 0, i.e., with no queries in hand.
3.3 Expert’s Perspective

We now turn to formulating the expert’s revenue as a function of the bundle characteristics \((k, c_\gamma^k)\) she sets. The expert’s revenue is derived in Section 2 by \(\pi = c_\gamma^k \eta_b - \eta_q d_c\). Therefore we need to formulate \(\eta_b\) and \(\eta_q\).

**Expected number of bundles purchased** The expected number of bundles purchased by the searcher is the expected number of times the searcher queries the expert when in state \(\gamma = 0\) (either transitioning to state \(\gamma = k - 1\) or terminating the search) (see Figure 1). In order to calculate \(\eta_b\), we compute the following probabilities:

<table>
<thead>
<tr>
<th>Represents</th>
<th>General formulation</th>
<th>HSGN formulation</th>
</tr>
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<tbody>
<tr>
<td>(P_{\gamma \rightarrow \gamma - 1}) Pr (querying and resuming, moving from state (\gamma &gt; 0) to (\gamma - 1))</td>
<td>(\int_{s_\gamma(V_{\gamma - 1}) &lt; V_{\gamma - 1}} f_s(V_{\gamma - 1}</td>
<td>s) ds)</td>
</tr>
<tr>
<td>(P_{\gamma \rightarrow \gamma}) Pr (resuming without querying, staying in state (\gamma &gt; 0))</td>
<td>(\int_{s_\gamma(V_{\gamma - 1}) &lt; V_{\gamma - 1}} f_s(V_{\gamma - 1}</td>
<td>s) ds)</td>
</tr>
<tr>
<td>(P_{\gamma \rightarrow \text{ter}}) Pr (querying and then terminating from state (\gamma))</td>
<td>(\int_{s_\gamma(V_{\gamma - 1}) &lt; V_{\gamma - 1}} f_s(V_{\gamma - 1}</td>
<td>s)(1 - F_v(V_{\gamma - 1}</td>
</tr>
<tr>
<td>(P_{0 \rightarrow k - 1}) Pr (querying and resuming, moving from state (\gamma = 0) to (k - 1))</td>
<td>(\int_{s_1(V_{k - 1}) &lt; V_{k - 1}} f_s(V_{k - 1}</td>
<td>s) ds)</td>
</tr>
<tr>
<td>(P_{0 \rightarrow 0}) Pr (resuming without querying when (\gamma = 0))</td>
<td>(\int_{s_1(V_{\gamma - 1}) &lt; V_{\gamma - 1}} f_s(V_{\gamma - 1}</td>
<td>s) ds)</td>
</tr>
<tr>
<td>(P_{0 \rightarrow \text{ter}}) Pr (terminating without querying when (\gamma = 0))</td>
<td>(\int_{s_1(V_{\gamma - 1}) &lt; V_{\gamma - 1}} f_s(V_{\gamma - 1}</td>
<td>s) ds)</td>
</tr>
<tr>
<td>(P_{\text{query}}) Pr (querying and terminating when (\gamma = 0))</td>
<td>(\int_{s_1(V_{\gamma - 1}) &lt; V_{\gamma - 1}} f_s(V_{\gamma - 1}</td>
<td>s)(1 - F_v(V_{\gamma - 1}</td>
</tr>
</tbody>
</table>

Notice that \(P_{\gamma \rightarrow \gamma - 1} + P_{\gamma \rightarrow \gamma} + P_{\gamma \rightarrow \text{ter}} = 1\), and similarly \(P_{0 \rightarrow k - 1} + P_{0 \rightarrow 0} + P_{\text{query}} + P_{0 \rightarrow \text{ter}} = 1\).

Let \(P_\gamma(T)\) denote the eventual probability of transitioning, when in state \(\gamma\), to state \(\gamma - 1\) (for \(\gamma = 0\) it is the probability of transition to state \(k - 1\)). Let \(P_{\text{cycle}}\) be the probability of starting at a given state and getting back to it after going through all other states (excluding search termination state). The values of \(P_\gamma(T)\) and \(P_{\text{cycle}}\) can be calculated as:

\[
P_\gamma(T) = \sum_{j=0}^{\infty} (P_{\gamma \rightarrow \gamma})^j P_{\gamma \rightarrow \gamma - 1} = \frac{P_{\gamma \rightarrow \gamma - 1}}{1 - P_{\gamma \rightarrow \gamma}} ; \quad P_{\text{cycle}} = \prod_{i=0}^{k-1} P_i(T)
\]

Now let \(P_\gamma(\text{Term})\), denote the probability of terminating the search in current state \(\gamma\) without transitioning to another state and \(P_0(\text{Term}|\text{Buy})\) denote the probability of eventually purchasing the bundle and terminating the search in state \(\gamma = 0\). These two probabilities are given by:

\[
P_\gamma(\text{Term}) = 1 - P_\gamma(T) = \frac{P_{\gamma \rightarrow \text{ter}}}{1 - P_{\gamma \rightarrow \gamma}}
\]
Using the above notation, \( \eta_b \) is given by:

\[
\eta_b = \sum_{j=1}^{\infty} j \Pr(j \text{ bundles are purchased})
\]

\[
= \sum_{j=1}^{\infty} j \Pr(j \text{ transitions from state 0 to } k - 1)
\]

\[
= \sum_{j=1}^{\infty} j P_{\text{cycle}}^{j-1} (P_0(\text{Term}|\text{Buy}) + P_0(T))(1 - P_{\text{cycle}})
\]

\[
= \sum_{j=1}^{\infty} j \left( \prod_{l=0}^{j-1} P_l(T) \right)^{j-1} (P_0(\text{Term}|\text{Buy}) + P_0(T))(1 - \left( \prod_{l=0}^{k-1} P_l(T) \right))
\]

**Expected number of queries used** The searcher may terminate the search before exhausting all the remaining queries purchased. This situation is unique to settings where the buyer (searcher in our case) is buying the right to use a service rather than actually receiving the service (or a product) upon purchase. For an expert offering a bundle \((k, c_k^e)\), the question of how many queries purchased are actually used is important when \(d_e > 0\). In order to calculate the expected number of queries used, \( \eta_q \), we first calculate the probability that exactly \( m \) queries are eventually used, denoted \( P_m(Q) \):

\[
P_m(Q) = \begin{cases} 
P_0(T) \prod_{j=0}^{j=k-1} P_j(T)P_{k-m+1}(\text{Term}) & m < k \\
P_0(T) \prod_{j=0}^{j=k-1} P_j(T)P_0(\text{Term}|\text{Buy}) + P_{\text{cycle}}P_0(\text{Term}|\neg\text{Buy}) & m = k \
\end{cases}
\]

For \( m > k \), we represent \( m \) as \( jk + i \) where \( i = m\%k \) and \( j = \lfloor \frac{m}{k} \rfloor \). Here, \( j \) represents the number of full cycles completed (when all queries in a bundle are used) and \( i \) represents the number of queries used prior to terminating before finishing the \( j + 1 \)st round. This cyclic nature gives us the following recurrence:

\[
P_{m=jk+i}(Q) = P_{\text{cycle}}P_{(j-1)k+i}(Q) = \cdots = P_{\text{cycle}}^i P_i(Q) = \left( \prod_{l=0}^{k-1} P_l(T) \right)^i P_i(Q)
\]

Therefore the expected number of queries is given by:

\[
\eta_q = \mathbb{E}(\text{Number of queries used}) = \sum_{m=0}^{\infty} mP_m(Q)
\]

\[
= \sum_{j=0}^{\infty} \sum_{i=1}^{k} (j + i)P_{\text{cycle}}^i P_i(Q) = \sum_{j=0}^{\infty} \sum_{i=1}^{k} (j + i) \left( \prod_{l=0}^{k-1} P_l(T) \right)^i P_i(Q)
\]

Once we have calculated \( \eta_q \) and \( \eta_b \), we can calculate the profit of an expert and find out the optimal strategy for an expert: \( \pi = \max_{c_k^e} (\eta_q c_k^e - \eta_b d_e) \).
Expected worth of an opportunity received

This quantity helps us analyze how much a searcher loses in search cost and query cost. It is also interesting to note how the search cost and query cost affects the expected worth of an opportunity received. Let $W$ be the random variable representing the worth of an opportunity received, then $E_i(W)$ represents the expected worth of the opportunity if the search terminates in state $i$ and $P_i(W)$ represents the probability of terminating at that state.

$$E_i(W) = \begin{cases} E(v|v > V_{i-1}, s > t_i) & i > 0 \\ E(v|v > V_{k-1}, t_k < s < t_i) Pr(v > V_{k-1}, t_k < s < t_i) & i = 0 \end{cases}$$

$$P_i(W) = \begin{cases} \sum_{j=0}^{\infty} P_{cycle} P_i(T) \left( \prod_{h=k-1}^{i} P_h(T) \right) P_i(\text{Term}) & i > 0 \\ \sum_{j=0}^{\infty} P_{cycle} (1 - P_i(T)) & i = 0 \end{cases}$$

$$E(W) = \sum_{i=0}^{\infty} E_i(W)P_i(W)$$

Expected number of searches

Let $\eta_s$ be the expected number of searches (opportunities examined by the searcher, $\eta_s \geq \eta_q$). We know the searcher’s utility, $V_0$, is the value of opportunity received minus the total search and query cost paid in the process ($V_0 = E(W) - \eta_s c_s - \eta_b c_k$). Therefore

$$\eta_s = \frac{E(W) - \eta_b c_k - V_0}{c_s}$$

3.4 Infinite Bundle Size

A specific interesting case to look at is whether the expert would ever want to sell an unlimited supply of services at a fixed price. This is equivalent to an infinite-sized bundle. In this case the model reduces to two states. The searcher starts with state $\gamma = 0$ and continues in this state until she either terminates search or transitions to state $\gamma = \infty$. In the latter case the searcher can keep querying the expert for any reasonable opportunity until she finally finds a sufficiently good opportunity and terminates search (the existence of a search cost ensures that this querying process does not go on forever). Being in state $\gamma = \infty$ is equivalent to being in the world of perfect signals. The optimal reservation value when in state $\gamma = \infty$ can thus be extracted from (e.g., [11]):

$$V_\infty = -c_s + V_\infty \int_0^{V_\infty} f_c(x) \, dx + \int_{V_\infty}^{\infty} xf_c(x) \, dx \tag{8}$$

For state $\gamma = 0$, we can use appropriate modifications of Equations 3 and 5-6, replacing $V_{k-1}$ with $V_\infty$ (realizing that the searcher transitions to state $\gamma = \infty$). The optimal strategy can be extracted from solving this set of the four equations.

4 Example

As can be observed from the analysis given in the former section, equilibrium in expert-mediated search with bundling derives from a complex set of dynamics.
in the system. The number of parameters affecting the equilibrium is substantial: the distribution of values, the correlation between signals and values, search frictions, the cost of querying the expert, and bundle size, all affect the overall outcome of the process. A static analysis, uncovering phenomenological properties of the model is therefore difficult and restricted. Instead, we turn to a specific example to outline some interesting effects of bundling in this domain.

We illustrate the optimal strategies for the searcher and the expert, assuming some specific distributions of the true values and signals. We consider a case where the signal is an upper bound on the true value. Going back to the used car example, sellers and dealers, offering cars for sale, usually make cosmetic improvements to the cars in question, and proceed to advertise them in the most appealing manner possible, hiding defects using temporary fixes. Specifically, following [4], we assume signals \( s \) are uniformly distributed on \([0, 1]\), and the conditional density of true values is linear on \([0, s]\). Then

\[
\begin{align*}
  f_s(s) &= \begin{cases} 
    1 & \text{if } 0 < s < 1 \\
    0 & \text{otherwise}
  \end{cases} \\
  f_v(y|s) &= \begin{cases} 
    \frac{2y}{s^2} & \text{for } 0 \leq y \leq s \\
    0 & \text{Otherwise}
  \end{cases}
\end{align*}
\]

We can use these distributions to solve the systems of equations as described in detail in the previous section. Solving these leads to several interesting insights on the effects of bundling. Surprisingly, we find that forced bundling can be pareto improving. It can lead to improvement not only in the profit of the monopolist expert, which is to be expected, but simultaneously to improvement in the expected utility of the searcher from engaging in the search process. Figure 3 shows the effect and the intuition. The figure depicts the expected utility to the buyer and the expert’s expected net profit (alongside the bundle and per-query costs) as a function of the bundle size \( k \) for \( c_s = 0.01 \) and different \( d_e \) values. When the marginal cost of producing an expert report is zero (as in the “digital services” case: an extra Carfax report can be produced essentially for free), there is no significant added cost to the expert. The search overall is becoming more efficient, and the expert and the buyer can split the additional utility. On the other hand, for non-zero marginal cost of producing an extra expert report, the benefit to the expert of selling higher bundle sizes rapidly declines after bundle sizes of two and three, because there is a real cost incurred in producing the additional reports that the searcher may demand.

Another interesting observation concerns the correlation between bundling and search cost (the cost of seeing each opportunity initially, \( c_s \)) faced by the user. In a world of high search costs, users do not expect to keep searching for more than a few opportunities, so they are unlikely to be willing to purchase a bundle of high size. Interestingly, when the search costs become very low, we find that it can be optimal for the expert to sell an unlimited subscription to her services for a fixed fee. Figure 4 shows an example of this phenomena in the zero marginal cost scenario. A search cost \( c_s = 0.001 \) yields a setting where it is optimal for the expert to offer infinite-size bundles, but increasing the search cost slightly to 0.003 leads to high (but far from infinite) bundle sizes being

\footnote{For each bundle size, the appropriate optimal \( c^k_e \) is used.}
Fig. 3. Effect of bundling on the price charged by the expert and its impact on the searcher's utility and the expert's profit. In this case, the searcher's utility increases as the bundle size increases. We see that although the overall bundle price, $c_k$, increases with the increase in bundle size, the cost per query decreases rapidly.

Both these observations, that increasing either the marginal cost of producing expert services and/or the cost incurred by the buyer in searching lead to smaller bundle sizes being preferred, correspond well to the real world. Unlimited subscription models are usually available in online services, where vanishing marginal costs and low search costs dominate (like autocheck.com), whereas traditional marketplaces have smaller bundle sizes.

5 Discussion

Experts can play a significant role in search-based marketplaces where there is a niche for information brokers – this is the premise of mechanics who inspect used cars to make sure they are not lemons and financial experts for due diligence of companies at the service of investors, and more. The ubiquity of electronic markets has changed the nature of many markets not just by lowering the cost of search to the consumer, but also by making it possible for experts to produce additional expert reports and communicate them to the consumer at
little to no extra cost. Expert services like Carfax have emerged naturally. An interesting observation is that these agencies have experimented with several different bundling and subscription models. In this paper we provide a model that helps us make sense of these different bundling strategies. While bundling has been studied in similar domains, e.g., Bakos and Brynjolfsson have shown that bundling is particularly useful for digital information goods because of the low marginal cost [2], prior research has not considered the interactions of bundling and search.

Surprisingly, we find that constraining experts to sell only bundles of their services can improve outcomes for both experts (who make higher profits) and searchers (who gain in expected utility of the search process). The intuition is that, by purchasing a bundle, the searcher is less constrained by the marginal cost of expert services and can exploit the search process to find a better opportunity. Another surprising result is that in some circumstances the expert may in fact maximize profit by selling an unlimited subscription to its service, compared with any finite bundle size. These circumstances are characterized by very low search costs and close-to-zero marginal costs of producing an extra unit of the expert’s service, a case which is highly applicable in electronic marketplaces.

Similar to the general bundling literature, we also observe negative correlation between marginal cost and optimal bundle size. In addition, we show that search cost can also affect optimal bundling significantly. Bundling is discouraged if the search cost is high, even if the marginal cost is zero, because buyers will typically not want to sample many opportunities, so the marginal benefit to them of extra expert reports that are essentially “free” is minimized.

There are three major directions for future work. First, our model only allows for bundles of fixed size. Mixed bundling, where the expert sells bundles of different possible sizes, has been shown to perform better in the theory of product bundling, and would be an interesting extension in our domain. Second, our model assumes a monopolistic expert. Bakos and Brynjolfsson have shown
that bundling, even in inferior digital "information" goods, with close to zero marginal cost has the potential to drive away superior quality sellers [3]. However, their analysis does not take search into consideration. Incorporating experts of different quality and cost who compete with each other in our model could reveal new insights. Third, in our relatively simple model, the improvement in utilities is essentially free and social-welfare maximizing, as it is a product of improving the efficiency of search. This is not unrealistic, and such "one-sided" search models have been show to have great applicability. However, in many real markets, it is also important to consider the utility of the seller, which involves modeling search as a two-sided process. Even in two-sided markets search frictions can play a major role, thus it is important to investigate whether the presence of experts can improve outcomes in such markets.

References