Dataflow Analysis

- Lattice theoretic foundations
  - Partial ordering
  - Meet, Join, Lattice, Chain
- Round robin fixed point iteration
- Function properties
  - Monotonicity
  - Distributivity
- Justification for using fixed-point iteration on dataflow equations
- Meet Over all Paths solution

Lattice Theory

- Partial ordering $\leq$
  - Relation between pairs of elements
  - Reflexive $x \leq x$
  - Anti-symmetric $x \leq y, y \leq x \Rightarrow x = y$
  - Transitive $x \leq y, y \leq z \Rightarrow x \leq z$
- Poset (Set S, $\leq$)
- 0 Element $0 \leq x, \forall x \in S$
- 1 Element $1 \geq \forall x \in S$

A poset need not have 0 or 1 element.

Lattice Theory

- **Greatest lower bound (glb)**

  $l1, l2 \in$ poset S, $a \in S$ is glb($l1$, $l2$)
  if $a \le l1$ and $a \le l2$ then
  for any $b \in S$, $b \le l1$, $b \le l2 \Rightarrow b \le a$

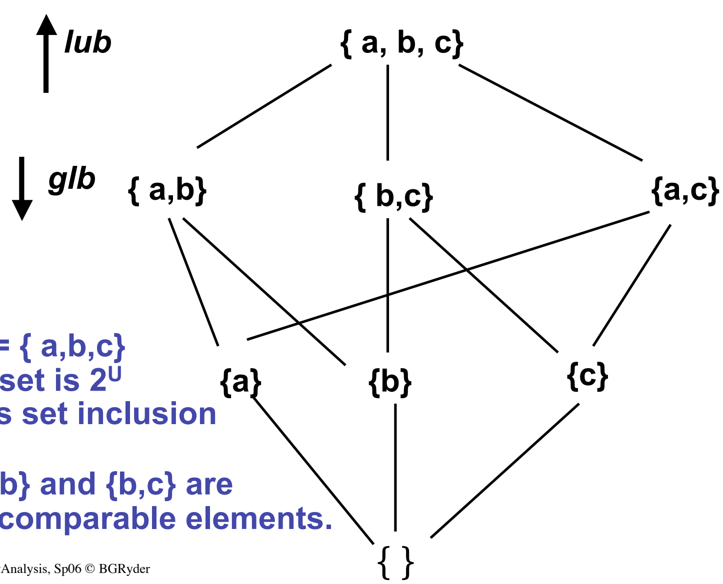
if glb is unique it is called the meet (Λ) of l1 and l2.

- **Least upper bound (lub)**

  $l1, l2 \in$ poset S, $c \in S$ is lub($l1$,$l2$)
  if $c \ge l1$ and $c \ge l2$ then
  for any $d \in S$, $d \ge l1$, $d \ge l2 \Rightarrow c \le d$.

if lub is unique is called the join (υ) of l1 and l2.

Poset Example



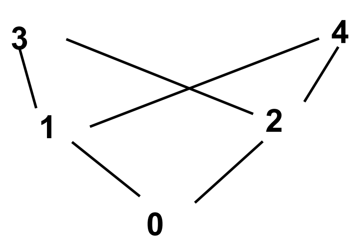
U = { a,b,c}
poset is 2^U
≤ is set inclusion

{a,b} and {b,c} are incomparable elements.

Definition of a Lattice (L, Λ, υ)

- L, a poset under ≤ such that every pair of elements has a unique glb (meet) and lub (join).
- A lattice need not contain an 0 or 1 element.
- A finite lattice must contain an 0 and 1 element.
- Not every poset is a lattice.
- If a ≤ x ∀ x ∈ L, then a is 0 element of L
- If x ≤ a ∀ x ∈ L, then a is 1 element of L

a poset, but not a lattice



There is no lub(3,4) in this poset so it is not a lattice.

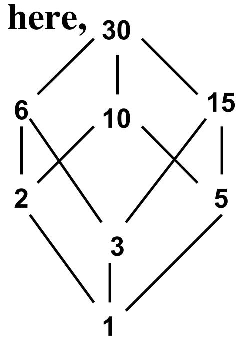
Examples of Lattices

- H = (2^U , ∩, ∪) where U is a finite set
  - glb (s1, s2) is (s1 Λ s2) which is s1 ∩ **s2**
  - **lub (s1, s2) is (s1** υ s2) which is s1 ∪ **s2**

- J = (N_1 , gcd, lowest common multiple)
  - partial order relation is integer divide on N_1
    n1 | n2 if division is even
  - lub (n1, n2) is n1 υ n2 = lowest common multiple(n1,n2)
  - glb (n1,n2) is n1 Λ n2 = greatest common divisor (n1,n2)

Chain

- A poset C where, for every pair of elements
c1, c2 ∈ C, either c1 ≤ c2 or c2 ≤ c1.
  e.g., { } ≤ {a} ≤ {a,b} ≤ {a,b,c}
  and from the lattice as shown here,
  1 ≤ 2 ≤ 6 ≤ 30
  1 ≤ 3 ≤ 15 ≤ 30



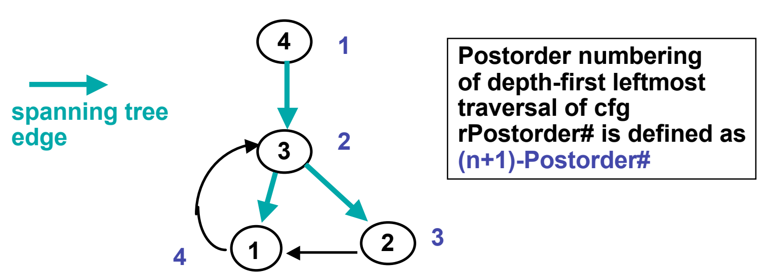
Lattices are used in dataflow analysis to argue the existence of a solution obtainable through fixed-point iteration.

Round-robin Fixed-point Iteration

- Iterates through nodes of cfg in specific node order, which preserves the partial order of the graph.
- Recall worklist algorithm(per bit) has O(me) work where |E|=e and m is number of dataflow facts in the set.
- Postorder on depth-first spanning tree defines rPostorder which is *reverse postorder*
  - Ancestor-first node ordering, good for forward dataflow problems
  - rPostorder_number =(| N |+1) - postorder_number
  - Postorder is good node ordering for backward dataflow problems

rPostorder Example

control flowgraph



Round-robin Iteration

Form depth-first spanning tree of control flowgraph and derive rPostorder numbering for nodes

change = true

Initialize REACH(m) = ∅, ∀ m

```
while (change) do
{   change = false
    for (m = 2; m <= n; m++)
    { new = ∪ ( Reach(j)  ∩  pres(j) ∪ dgen(j) )
          j = pred(m)

      if (new != REACH(m))
                then {REACH(m) = new;  change = true}
    }
}
```

Round-robin tends to push dataflow facts as deep as possible on acyclic paths in the flowgraph on each iteration

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```

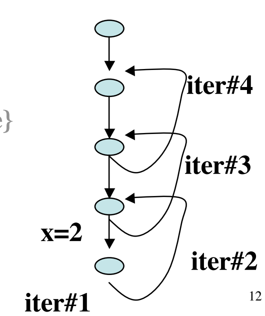
O(n)

#iters of while * cost each iteration

cost *for* statement ~ cost of all ops during lifetime of *for*

d = 3; d+2 = 5

iter#5, no change



Cost of Loop Operations

Calculate *for* statement cost over all its iterations:

1. 2(n-1) bit vector operations to calculate each node's factor new = ∪ (Reach(j) ∩ pres(j) ∪ dgen(j) and memoize it;
2. unions: (#preds(m)-1) for each node m means we do (e - n) unions per *for* statement for all nodes;
3. rest of *for* is constant number of operations

Each while loop iteration costs $(e - n) + (2n-2) \approx e + n$;

Therefore, entire algorithm costs O(de), ($\approx$ O(dn) by assumption for flowgraphs), bit vector operations where d+2 for classical bitvector problems.

Note: this dominates O(e) of calculation of rPostorder numbers and the O(n) of initialization loop.

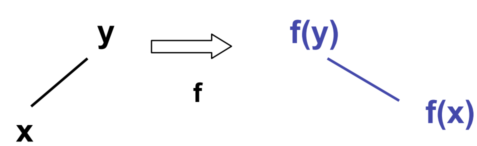
More Lattice Properties

- *Finite length*: if every chain in lattice is finite
- *Bounded lattice*: if lattice contains both 0 and 1 elements
- *Distributive lattice*: if $\forall$ x,y,z $\in$ S

  $x \upsilon (y \Lambda z) = (x \upsilon y) \Lambda (x \upsilon z)$

  $x \Lambda (y \upsilon z) = (x \Lambda y) \upsilon (x \Lambda z)$
- $(S,\leq)$ poset, f: S --> S is monotonic iff

  $\forall x, y \in S,\ x \leq y \Rightarrow f(x) \leq f(y)$
- Monotonic functions preserve domain ordering in their range values



Fixed point theorem - Why it works?

Intutition--

Given a 0 in lattice and monotonic function f, $0 \leq f(0)$.

Apply f again and obtain

$0 \leq f(0) \leq f(f(0)) = f^2(0)$

Continuing,

$0 \leq f(0) \leq f^2(0) \leq f^3(0) \leq \ldots \leq f^k(0) = f^{k+1}(0)$ for a finite chain lattice.

This is tantamount to saying $\lim_{k \Rightarrow \infty} f^k(0)$ exists and is called the *least fixed point* of f,

since $f(f^k(0)) = f^k(0)$

Fixed Point Theorem

Thm: f: S --> S monotonic function on poset (S, ≤) with a 0 element and finite length. The *least fixed point* of f is $f^k(0)$ where

i. $f^0(x) = x$,

ii. $f^{i+1}(x) = f(f^i(x)), i \geq 0$,

iii. $f^k(0) = f(f^k(0))$ and this is the smallest k for which this is true.

- For any p such that f(p)=p, $f^k(0) \leq p$.
- Theorem justifies the iterative algorithm for global data flow analysis for lattices & functions with right properties
- Dual theorem exists for 1 element and *maximal fixed point* for k such that $f^k(1) = f^{k+1}(1)$.

Application to DFA on CFG

- **Cartesian cross product of posets is a poset**
  - **if (S, ≤) is a poset, then (S x S x ... x S, ≤') is a poset whose partial ordering is component-wise ≤**

  W=(W1,W2,…,Wn) ≤'Y=(Y1,Y2,…,Yn) iff

  W1 ≤ Y1 and W2 ≤ Y2, …, and Wn ≤ Yn
- **Cartesian cross product of lattices is a lattice (with an induced partial ordering)**
- **Monotone function on cross product lattice can be built from monotone functions on each component lattice** **F(Y1, Y2,..., Yn) = (g_1 (Y1, Y2,...,Yn),..., g_n(Y1, Y2,...,Yn))**

where g_I : (L,L, ..., L) --> L and ≤' is component-wise ≤ from L

Example - Available Exprs

- **lattice is 2^{Exprs} where Exprs is set of all binary expressions in program**
- **Partial order is ⊆ (subset inclusion) so meet is ∩**
- **< Exprs,Exprs,...,Exprs> is 1 element**
- **< ∅, ∅,..., ∅) is 0 element**
- **From the data flow equations for AVAIL, we know that if a set of dataflow facts X is true on entry to a flowgraph node n, then f(X) is true on each exit edge of n where**

  f(X) = epres(n) ∩ X ∪ egen(n)

f is called the *transfer function* for AVAIL

Example, Available Exprs

- **Cross product lattice is**
  - **(2^{Exprs} , 2^{Exprs} ,..., 2^{Exprs}) with n components where n is number of nodes in the cfg and ≤' is ≤ component-wise**
- **Avail equation at a node can be expressed thusly,**
  Avail(j) = $\bigcap_{m \in Pred(j)}$ { Avail(m) ∩ epres(m) ∪ egen(m) }
  - **AVAIL (j) is the solution at entry of node j and f(AVAIL(j)) is solution at exit of node j,**

  $g_j = \cap\, f(g_m)$, $m \in Pred(j)$
  - **One step of the worklist algorithm maps a potential solution for AVAIL at a node of the cfg to another potential solution for that node**

Example, Available Exprs

- **Can you show g_i monotone?**

  $g_i : (2^{Exprs} , 2^{Exprs} ,...., 2^{Exprs}) \text{-->} 2^{Exprs}$
- **Then this induces the monotonicity of F,**

  $F = (g_1,..., g_n)$
- **Application of dual of fixed point theorem here to find the maximal fixed point. Iterate down from the 1 element.**
  - **Initialize ρ to ∅, all other cfg nodes to Exprs.**

Sketch of Validity Proof of Iterative Algorithm for DFA

- Let Redef $(Y_1^i \ldots Y_n^{\ i})$ be result after i steps of the worklist algorithm (per node) for DFA.

  Redef$(Y_1^i \ldots Y_n^{\ i}) = (Y_1^i \ldots Y_{k-1}^{\ i},\ g_k\,(Y_1^i \ldots Y_n^{\ i}), Y_{k+1}, \ldots Y_n)$
- In the next iteration (i+1st),only 1 component of $Y= (Y_1^i \ldots Y_n^{\ i})$ changes and this component is chosen non-deterministically.
- Recall we have a function defined on the flowgraph: $F(Y_1,\ldots,Y_n) = (g_1\,(Y_1,\ldots,Y_n),\ldots, g_n(Y_1,\ldots,Y_n))$ where g_j : (L,L, ..., L) --> L is defined by the dataflow problem

Sketch of Proof, cont.

- Assume minimal fixed point of F is $F^m(0)$.
- Must show iterative algorithm halts at a fixed point of Redef function, Redefmin, and that this fixed point is meaningful to the data flow problem under solution, namely Redefmin = $F^m(0)$.
- First, show iterates of Redef form a chain.

  $0 \leq' Y^1 \leq' Y^2 \leq' Y^3 \leq' \ldots \leq' Y^k$

Therefore, Redefmin = Redef^k(0) for some smallest k, by finiteness of chains on the lattice.

Sketch of Proof, cont.

Second, let $F^m(0)$ be the minimum fixed point of the data flow function F on the cross product lattice. Show $\boxed{F^m(0) \leq' \text{Redefmin}}$.

$F^m(0) \leq'$ Redefmin because Redefmin is also a fixed point of F, but not necessarily its minimal fixed point.

Third, show Redefmin $\leq'$ $F^m(0)$ in two steps

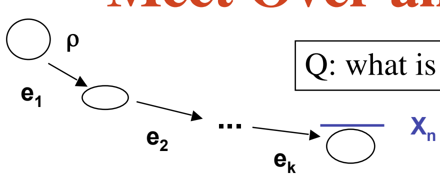
(1) $\text{Redef}^k(0) \leq' F^k(0)$ for all k and

(2) Redefmin = $\text{Redef}^k(0)$ for large enough k,

so for $k \geq k'$ large enough, $\boxed{\text{Redefmin} \leq' F^{k'}(0) \leq' F^m(0)}$ by defn of minimum fixed pt

Therefore, Redefmin= $F^m(0)$ and the fixed point iterative procedure does find the minimal fixed point on the lattice

Meet Over all Paths solution



Q: what is relation between MFP and MOP?

- Maximal data flow information desired is obtained by traversing ALL PATHS from ρ to n.
- Consider $X_n = \Lambda f_Q(0)$, for Q a path in the flowgraph, $Q = e_1, e_2, \ldots e_k$ so that

$$X_n = \underset{Q = e_k \ldots e_1}{\Lambda} f_{e_k} \circ f_{e_{k-1}} \circ \ldots \circ f_{e_2} \circ f_{e_1}(0)$$

MOP is best summary of data flow facts possible to compute at compile-time.