General Topology

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March 19, 2018

Basic Definitions

A topology is a pair \((X, \mathcal{T})\) where \(X\) is a set and \(\mathcal{T}\) is a set of subsets of \(X\) such that \(\mathcal{T}\) satisfies three properties:

- \(X \in \mathcal{T}\) and \(\emptyset \in \mathcal{T}\).
- \(\mathcal{T}\) is closed under arbitrary unions. I.e., if some (possibly uncountable) family of sets \(\{U_\alpha\} \in \mathcal{T}\), then \(\bigcup U_\alpha \in \mathcal{T}\).
- \(\mathcal{T}\) is closed under finite intersections. I.e., if \(U, V \in \mathcal{T}\), then \(U \cap V \in \mathcal{T}\).

The elements of \(\mathcal{T}\) are called open sets under the topology \((X, \mathcal{T})\), and the complements of open sets are called closed sets under \((X, \mathcal{T})\). A set is clopen if it is both open and closed (note, \(\emptyset\) and \(X\) are always clopen by definition).

An open set containing some \(x \in X\) is called an open neighborhood of \(x\). The largest open set contained in a set \(E\) is called the interior of \(E\) and is denoted \(E^\circ\). The smallest closed set containing \(E\) is called the closure of \(E\) and is denoted \(\overline{E}\). The boundary of a set \(E\), denoted \(\partial E\), is the set difference between the closure of \(E\) and the interior of \(E\). A point is on the boundary of \(E\) if and only if every open neighborhood of \(x\) contains points both in \(E\) and in \(E^c\). A set \(E\) is dense in \(X\) if \(\overline{E} = X\), and nowhere dense if \((\overline{E})^\circ = \emptyset\).

A sequence of points \(\{x_n\}\) in \(X\) is said to converge to a point \(x \in X\) if for every open neighborhood \(U\) of \(x\), there exists some value \(N\) such that \(x_n \in U\) for all \(n > N\). Then \(x\) is said to be a limit point of \(\{x_n\}\). In general, the limit point of a sequence need not be unique. However, in a Hausdorff space (described later) every limit is unique. Using this terminology, the following are equivalent definitions of closedness:

- A set \(C\) is closed (by definition) if it is the complement of an open set.
- A set \(C\) is closed if and only if every convergent sequence of points in \(C\) converges to a point in \(C\) (i.e., \(C\) contains all its limit points).
- A set \(C\) is closed if and only if it contains all its boundary points.

A topology \(\mathcal{T}\) is said to be generated by a basis \(\mathcal{B}\) if all the elements of \(\mathcal{T}\) can be written as unions of elements in \(\mathcal{B}\). The standard topology on \(\mathbb{R}^d\) is the topology generated by open balls \(B(x, r)\) where \(B(x, r) = \{y : d(y, x) < r\}\) where \(d(x, y) = \|x - y\|_2\). In general, any topology generated by open balls with respect to some metric \(d\) is called a metric space. A metric
space $X$ is said to be complete if every Cauchy sequence: $(\{x_n\}$ such that $d(x_n, x_m) \to 0$ as $n, m \to \infty$) converges to a point in $X$.

A set $K$ is compact if every open cover of $K$ (i.e., set of open sets whose union contains $K$) has a finite subcover. The Heine-Borel Theorem states that in a finite-dimensional metric space, this is equivalent to stating that $K$ is closed and bounded. In an infinite-dimensional metric space, we need the further condition that $K$ is closed and bounded, and $K$ lies within some the $\varepsilon$-neighborhood of some finite-dimensional metric space where $\varepsilon > 0$. An entire space $X$ is compact if as a set, $X$ is compact under its own topology.

**Continuity and Homeomorphisms**

A function $f : X \to Y$ is said to be continuous with respect to $(X, T_X)$ and $(Y, T_Y)$ if $f^{-1}(U) \in T_X$ for all in $U \in T_Y$.

Note, that if we were to rename the elements of $X$ while maintaining the same topology on the renamed set, then the change would be superficial and, from a topological perspective, we wouldn’t be able to distinguish between the spaces. For example,

$$X = \{a, b, c\}, \quad T_X = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$$

is indistinguishable from

$$Y = \{1, 2, 3\}, \quad T_Y = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}.$$ 

We capture this notion of sameness by saying $X$ and $Y$ are homeomorphic. If a continuous bijective function $h$ exists between $X$ and $Y$, then clearly they are homeomorphic, and we call $h$ a homeomorphism between $X$ and $Y$. Properties of a topology that are preserved under homeomorphisms are called topological invariants. For example, the existence of clopen sets other than $X$ and $\emptyset$ is a topological invariant.

One important topological invariant is the Hausdorff property, which states that for every pair of points $x_1, x_2 \in X$, there exists a neighborhood $U_1$ of $x_1$ and $U_2$ of $x_2$ such that $U_1 \cap U_2 = \emptyset$. Another important topological invariant is separability, defined as the existence of a countable dense subset of $X$. For example, $\mathbb{R}$ is separable under the standard topology since the rationals are countable and dense in $\mathbb{R}$.

**Manifolds**

A $n$-dimensional manifold is a topological space that “looks like” $n$-dimensional Euclidean space when we zoom in close enough. To put this formally, every point $x$ in a manifold has an open neighborhood $U_x$ that is homeomorphic to $\mathbb{E}^n$ (that is, if we restrict the topology on $X$ to its intersection with $U_x$). For example, the surface of the Earth is a 2-dimensional manifold because to a person standing on its surface, it looks like a flat plane.