CS 4204 Computer Graphics

2D Transformations

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References:
"Introduction to Computer Graphics" course notes by Doug Bowman
Interactive Computer Graphics, Fourth Edition, Ed Angle

Transformations

What are they?

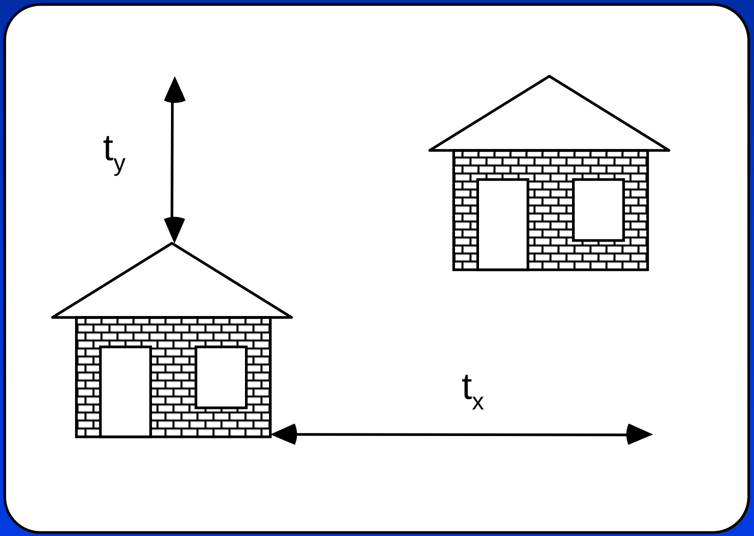
- changing something to something else via rules
- mathematics: mapping between values in a range set and domain set (function/relation)
- geometric: translate, rotate, scale, shear,…

Why are they important to graphics?

- moving objects on screen / in space
- mapping from model space to world space to camera space to screen space
- specifying parent/child relationships
- …

Translation

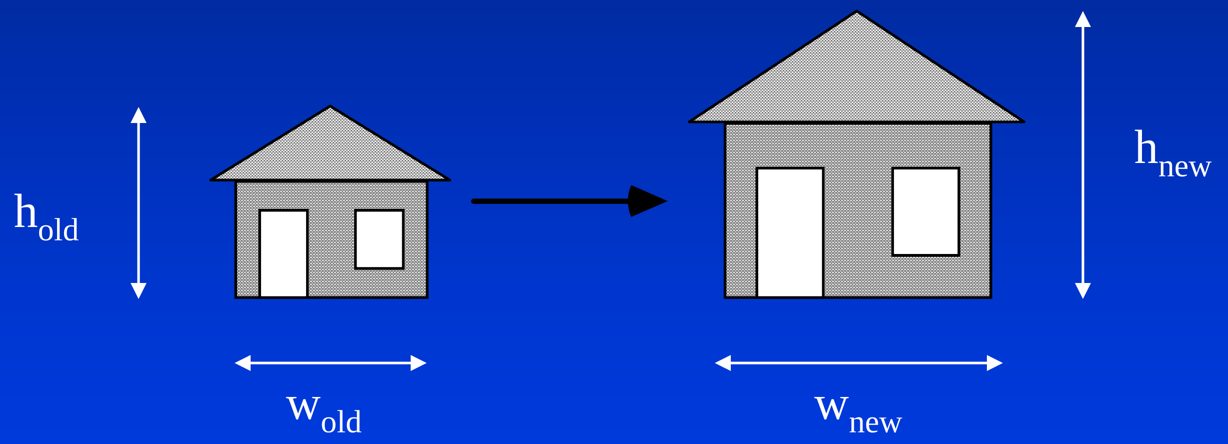
Moving an object is called a translation. We translate a point by adding to the x and y coordinates, respectively, the amount the point should be shifted in the x and y directions. <u>We translate an object by translating each vertex in the object.</u>



$$x_{new} = x_{old} + t_x;\ y_{new} = y_{old} + t_y$$

Scaling

Changing the size of an object is called a scale. We scale an object by scaling the x and y coordinates of each vertex in the object.



$$s_x = w_{new}/w_{old} \qquad s_y = h_{new}/h_{old}$$

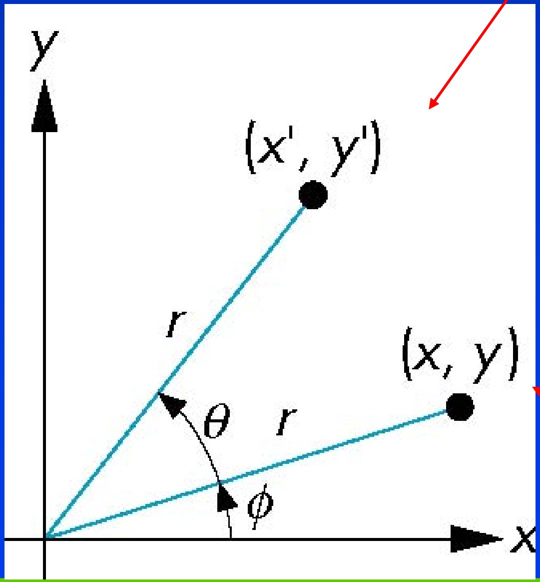
$$x_{new} = s_x x_{old} \qquad y_{new} = s_y y_{old}$$

Rotation about the origin

Consider rotation about the origin by q degrees

- radius stays the same, angle increases by *q*

$$x = r \cos (\phi + \theta)$$
$$y = r \sin (\phi + \theta)$$



$$x' = x \cos \theta - y \sin \theta$$
$$y' = x \sin \theta + y \cos \theta$$

$$x = r \cos \phi$$
$$y = r \sin \phi$$

Transformations as matrices

Scale:

$x_{new} = s_x x_{old}$

$y_{new} = s_y y_{old}$

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \end{bmatrix}$$

Rotation:

$x_2 = x_1 \cos\theta - y_1 \sin\theta$

$y_2 = x_1 \sin\theta + y_1 \cos\theta$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

Translation:

$x_{new} = x_{old} + t_x$

$y_{new} = y_{old} + t_y$

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix}$$

Homogeneous Coordinates

In order to represent a translation as a matrix multiplication operation we use 3 x 3 matrices and pad our points to become 3 x 1 matrices. This coordinate system (using three values to represent a 2D point) is called homogeneous coordinates.

$$P_{(x,y)} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_{x,y} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T_{x,y} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Composite Transformations

Suppose we wished to perform multiple transformations on a point:

$$P_2 = T_{3,1}P_1$$

$$P_3 = S_{2,2}P_2$$

$$P_4 = R_{30}P_3$$

$$M = R_{30}S_{2,2}T_{3,1}$$

$$P_4 = MP_1$$

Remember:

- Matrix multiplication is associative, not commutative!
- Transform matrices must be pre-multiplied
- The first transformation you want to perform will be at the far right, just before the point

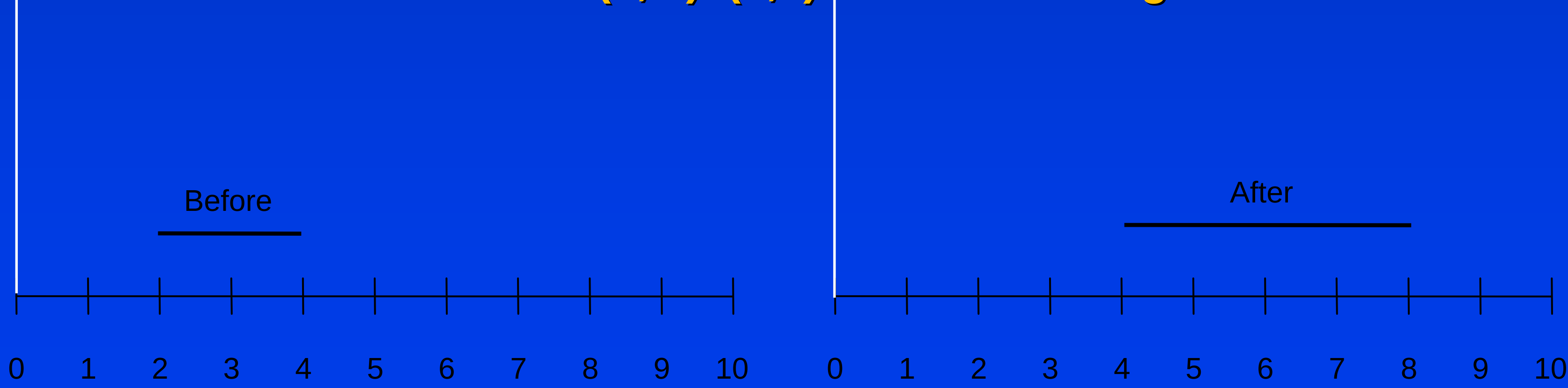
Composite Transformations - Scaling

Given our three basic transformations we can create other transformations.

Scaling with a fixed point

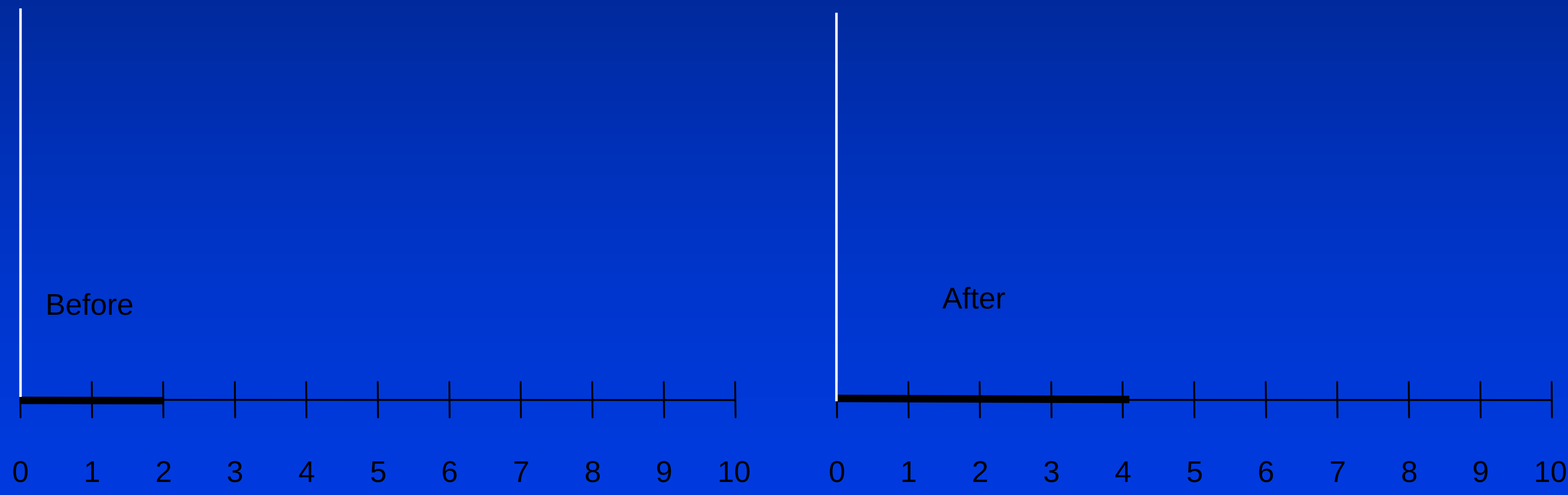
A problem with the scale transformation is that it also moves the object being scaled.

Scale a line between (2, 1) (4,1) to twice its length.



Composite Transforms - Scaling (cont.)

If we scale a line between (0,0) & (2,0) to twice its length, the left-hand endpoint does not move.



(0,0) is known as a ***fixed point*** for the basic scaling transformation. We can use composite transformations to create a scale transformation with different fixed points.

Fixed Point Scaling

Scale by 2 with fixed point = (2,1)

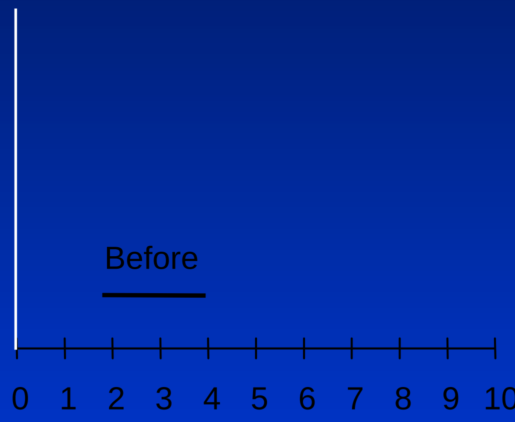
Translate the point (2,1) to the origin

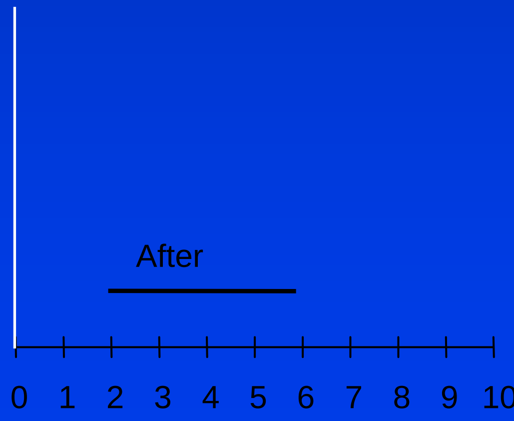
Scale by 2

Translate origin to point (2,1)

$$\underset{T_{2,1}}{\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}}\underset{S_{2,1}}{\begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}\underset{T_{-2,-1}}{\begin{bmatrix}1 & 0 & -2\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}} = \underset{C}{\begin{bmatrix}2 & 0 & -2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}$$

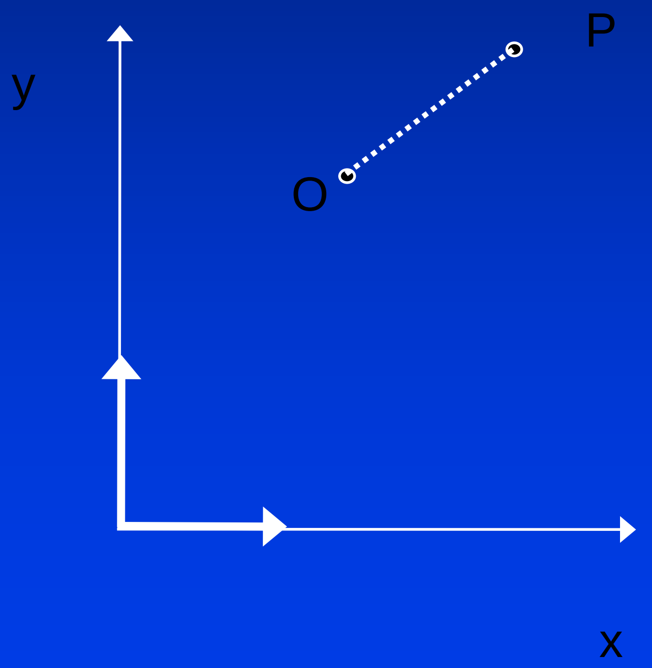
$$\underset{C}{\begin{bmatrix}2 & 0 & -2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix} \qquad \underset{C}{\begin{bmatrix}2 & 0 & -2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}\begin{bmatrix}4\\1\\1\end{bmatrix} = \begin{bmatrix}6\\1\\1\end{bmatrix}$$



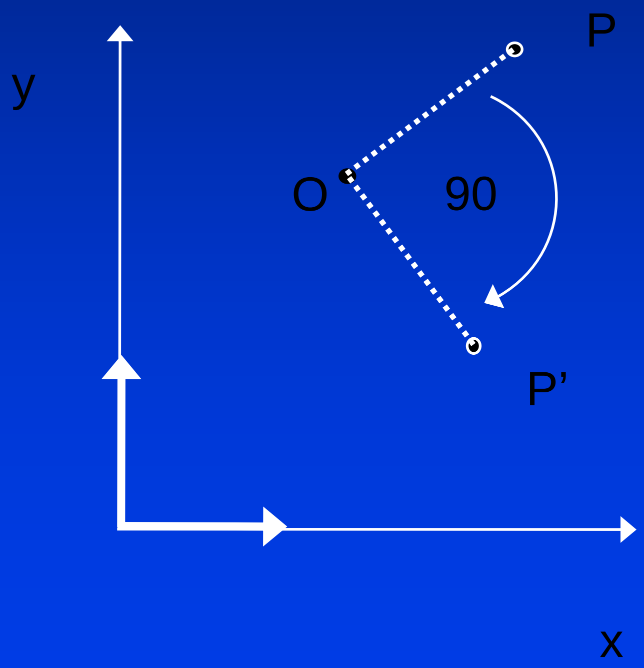


Example of 2D transformation

Rotate around an arbitraty point O:

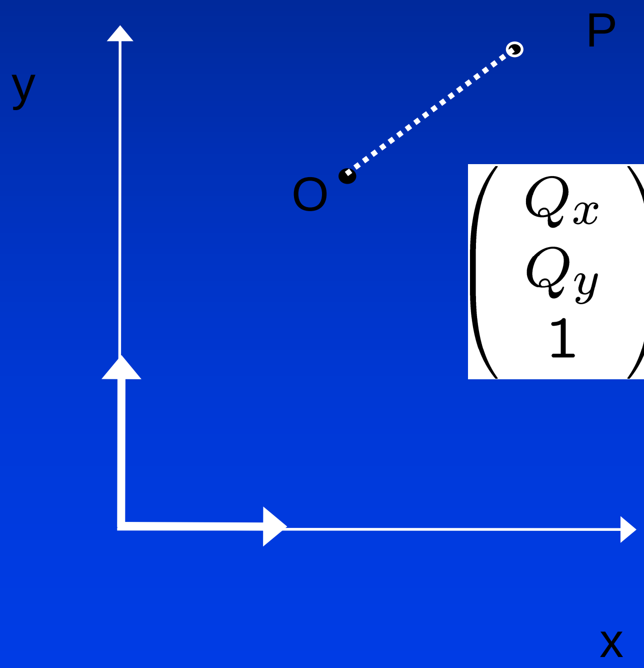


Rotate around an arbitraty point



Rotate around an arbitraty point

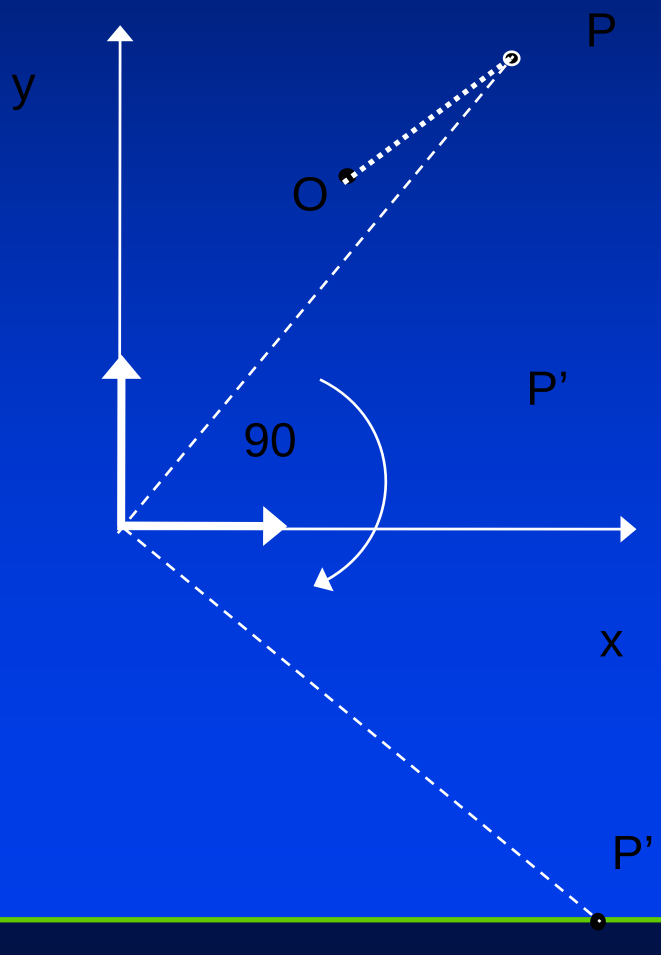
We know how to rotate around the origin



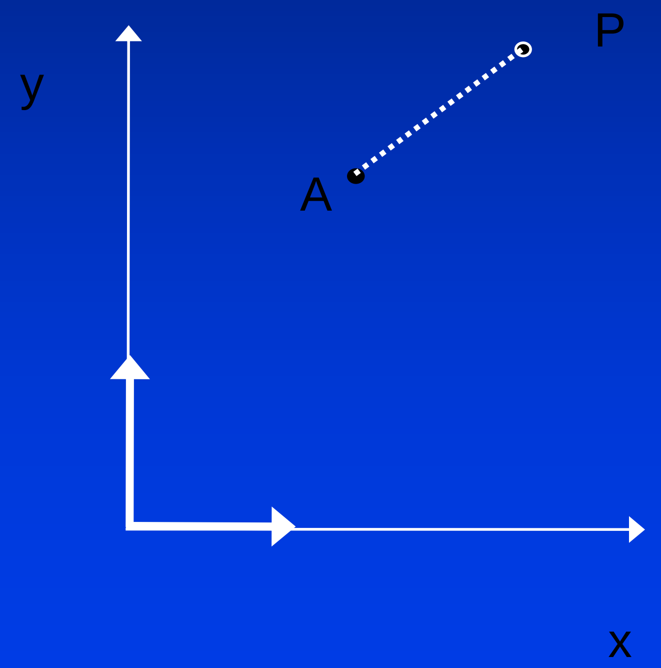
$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

Rotate around an arbitraty point

…but that is not what we want to do!

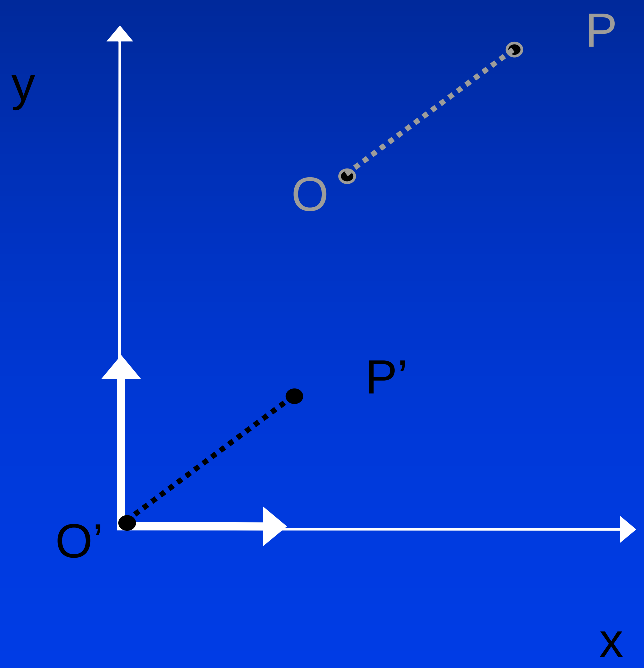


So what do we do?



Transform it to a known case

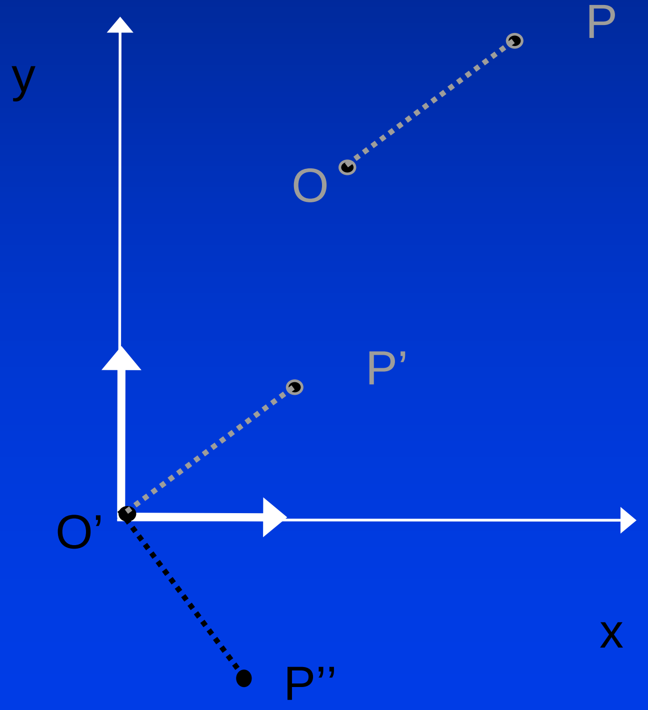
Translate(-Ox,-Oy)



Second step: Rotation

Translate(-Ox,-Oy)

Rotate(-90)

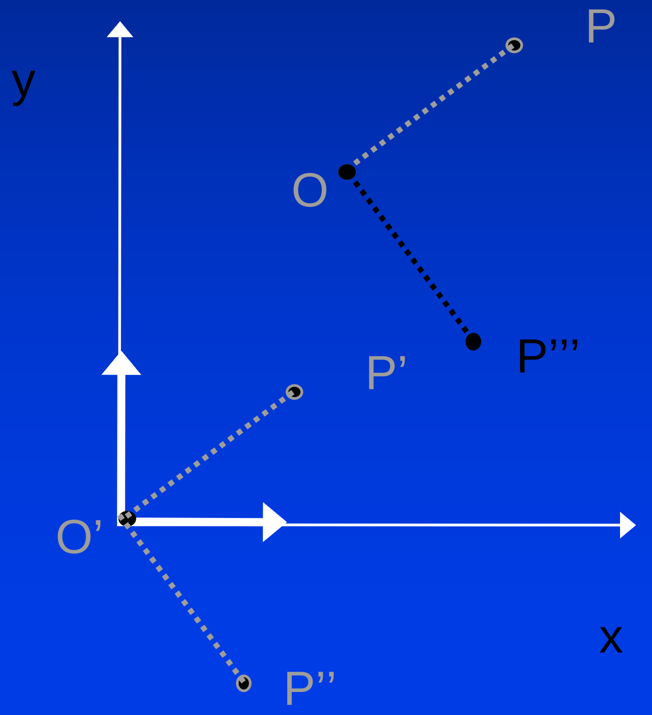


Final: Put everything back

Translate(-Ox,-Oy)

Rotate(90)

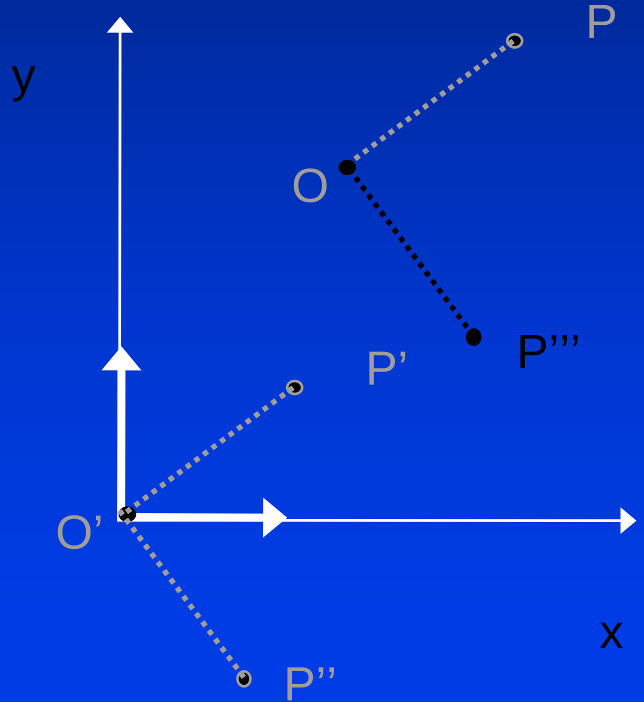
Translate(Ox,Oy)



Rotation about arbitrary point

IMPORTANT!: Order

M = T(Ox,Oy)R(-90)T(-Ox,-Oy)



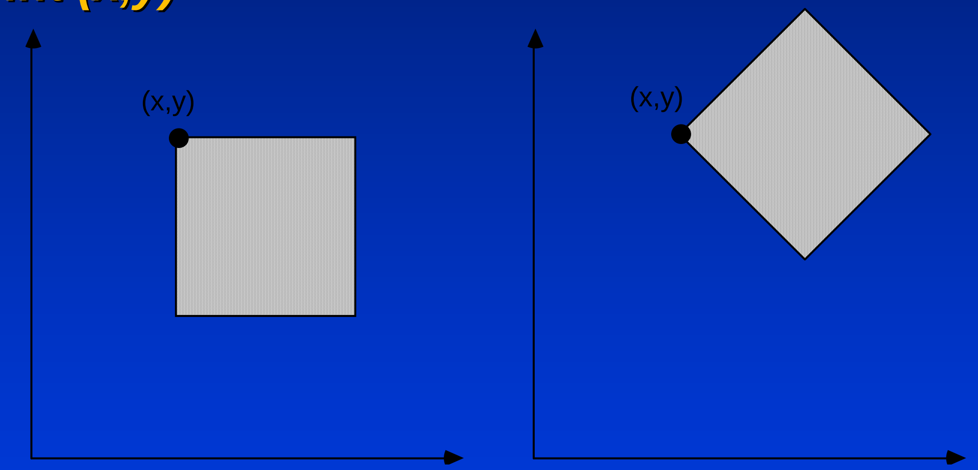
Rotation about arbitrary point

Rotation Of θ Degrees About Point (x,y)

Translate (x,y) to origin

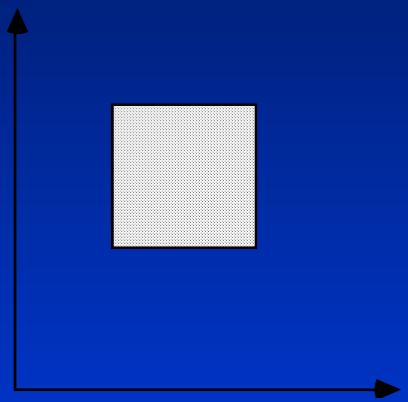
Rotate

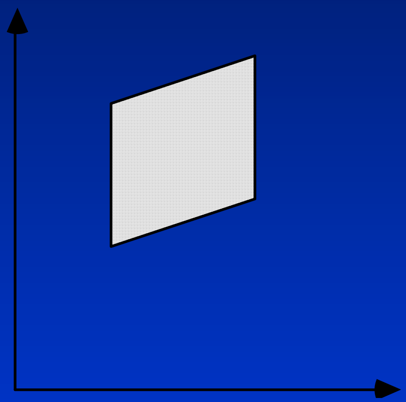
Translate origin to (x,y)

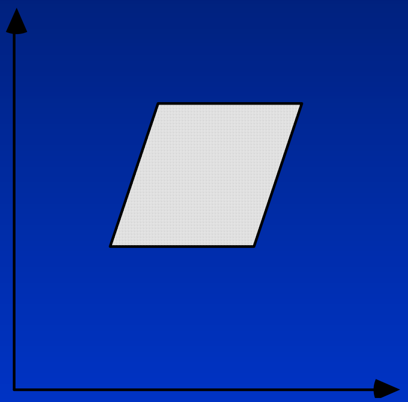


$$C = \underbrace{\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}}_{T_{x,y}} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_\theta} \underbrace{\begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}}_{T_{-x,-y}}$$

Shears







Original Data

y Shear

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

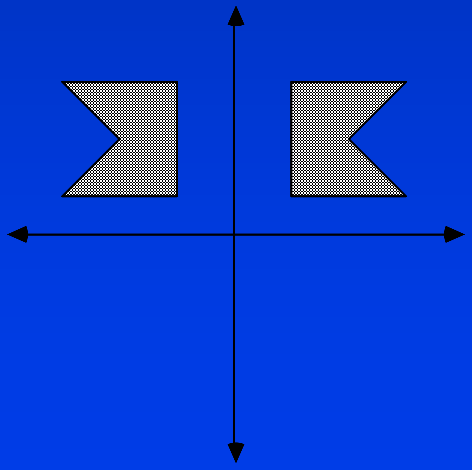
x Shear

$$\begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflections

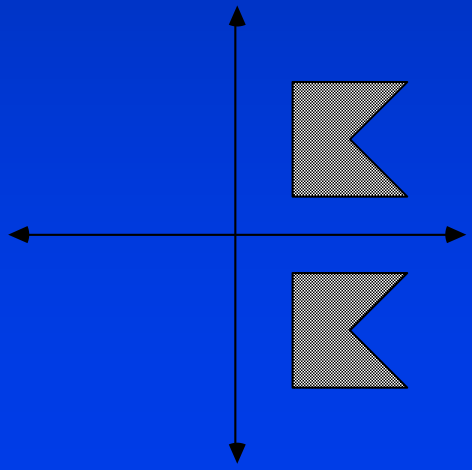
Reflection about the y-axis

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Reflection about the x-axis

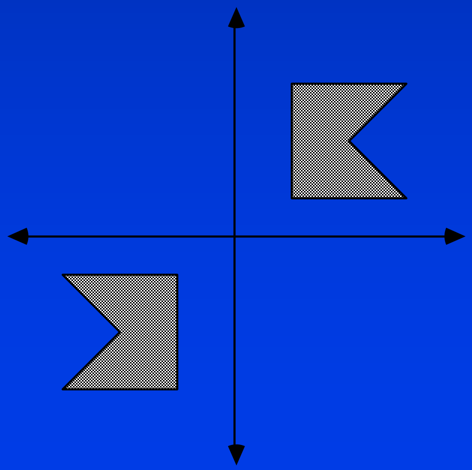
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



More Reflections

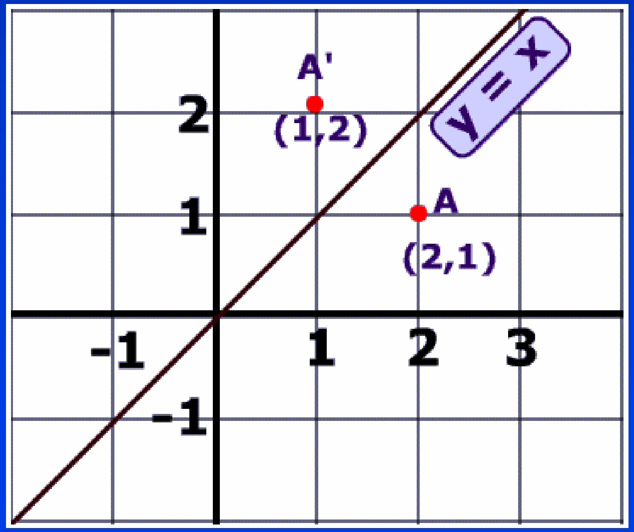
Reflection about the origin

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Reflection about the line y=x

?

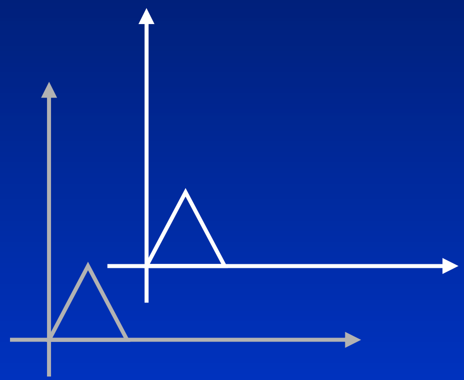


Transformations as a change in coordinate system

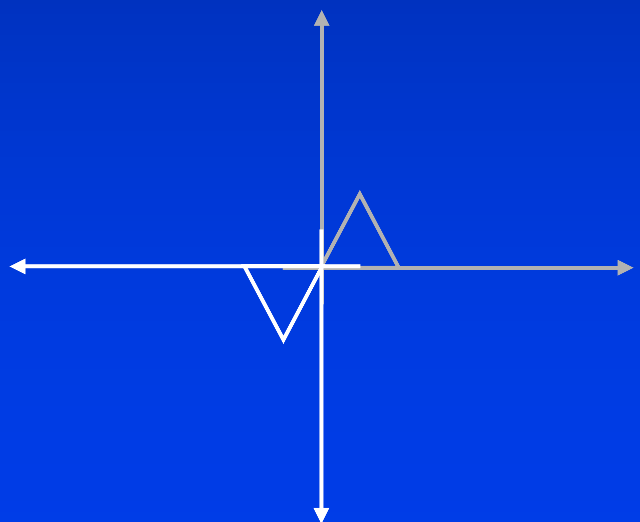
All transformations we have looked at involve transforming points in a fixed coordinate system (CS).

Can also think of them as a transformation of the CS itself

Transforming the CS - examples



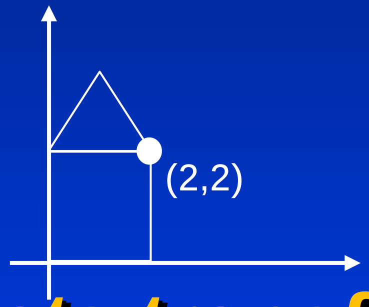
Translate(4,4)



Rotate(180°)

Why transform the CS?

Objects often defined in a "natural" or "convenient" CS

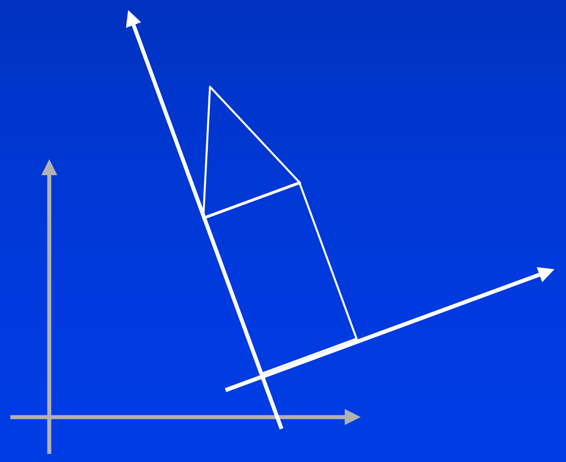


To draw objects transformed by T, we could:

- Transform each vertex by T, then draw
- Or, draw vertices <u>in a transformed CS</u>

Drawing in transformed CS

Tell system once how to draw the object, then draw in a transformed CS to transform the object



House drawn in a CS that's been translated, rotated, and scaled

$$M = S_{x,y} R_d T_{x,y}$$

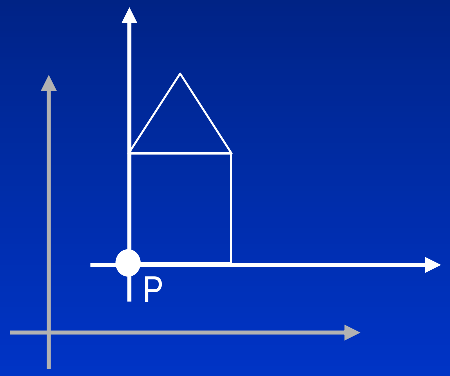
Mapping between systems

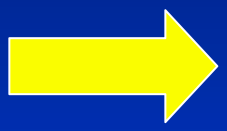
Given:

- The vertices of an object in CS_2
- A transformation matrix M that transforms CS_1 to CS_2

What are the coordinates of the object's vertices in CS_1?

Mapping example





Translate(4,4)

Point P is at (0,0) in the transformed CS (CS_2). Where is it in CS_1?

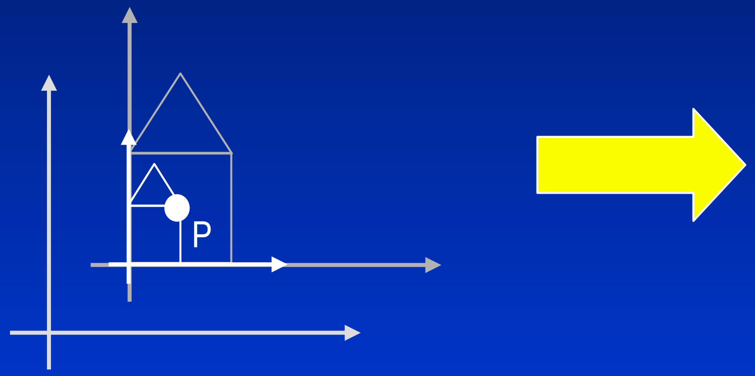
Answer: (4,4)

*Note: $(4,4) = T_{4,4}\ P$

Mapping rule

In general, if CS_1 is transformed by a matrix M to form CS_2, a point P in CS_2 is represented by MP in CS_1

Another example



Translate(4,4), then Scale(0.5, 0.5)

Where is P in CS_3? (2,2)
Where is P in CS_2? $S_{0.5,0.5}$ (2,2) = (1,1)
Where is P in CS_1? $T_{4,4}$ (1,1) = (5,5)

*Note: to go directly from CS_3 to CS_1 we can calculate $T_{4,4}$ $S_{0.5,0.5}$ (2,2) = (5,5)

General mapping rule

If CS_1 is transformed consecutively by M_1, M_2, …, M_n to form CS_{n+1}, then a point P in CS_{n+1} is represented by

$$M_1 M_2 \ldots M_n P \text{ in } CS_1.$$

To form the composite transformation between CSs, you postmultiply each successive transformation matrix.

OpenGL Transformations

Learn how to carry out transformations in OpenGL

- Rotation
- Translation
- Scaling

Introduce OpenGL matrix modes

- Model-view
- Projection

OpenGL Matrices

In OpenGL matrices are part of the state

Multiple types

- Model-View (**`GL_MODELVIEW`**)
- Projection (**`GL_PROJECTION`**)
- Texture (**`GL_TEXTURE`**) (ignore for now)
- Color(**`GL_COLOR`**) (ignore for now)

Single set of functions for manipulation

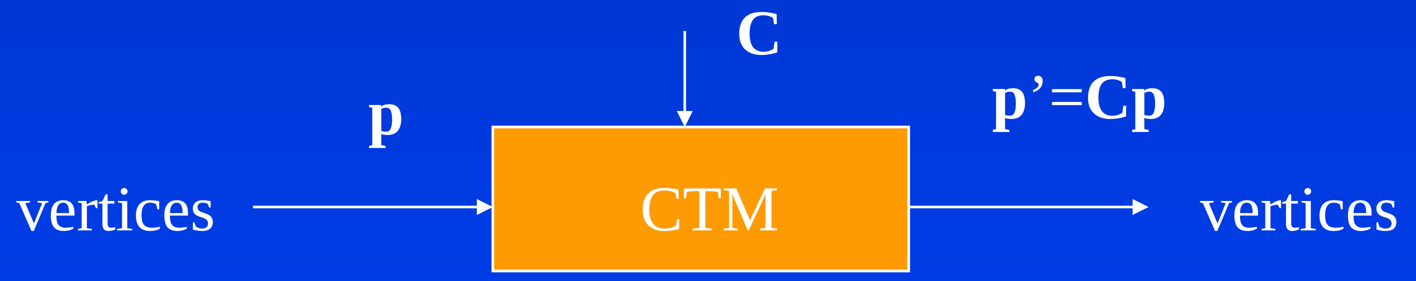
Select which to manipulated by

- **`glMatrixMode(GL_MODELVIEW);`**
- **`glMatrixMode(GL_PROJECTION);`**

Current Transformation Matrix (CTM)

Conceptually there is a 4 x 4 homogeneous coordinate matrix, the **current transformation matrix** *(CTM) that is part of the state and is applied to all vertices that pass down the pipeline*

The CTM is defined in the user program and loaded into a transformation unit



CTM operations

The CTM can be altered either by loading a new CTM or by postmutiplication

Load an identity matrix: $\mathbf{C} \leftarrow \mathbf{I}$
Load an arbitrary matrix: $\mathbf{C} \leftarrow \mathbf{M}$

Load a translation matrix: $\mathbf{C} \leftarrow \mathbf{T}$
Load a rotation matrix: $\mathbf{C} \leftarrow \mathbf{R}$
Load a scaling matrix: $\mathbf{C} \leftarrow \mathbf{S}$

Postmultiply by an arbitrary matrix: $\mathbf{C} \leftarrow \mathbf{CM}$
Postmultiply by a translation matrix: $\mathbf{C} \leftarrow \mathbf{CT}$
Postmultiply by a rotation matrix: $\mathbf{C} \leftarrow \mathbf{C\,R}$
Postmultiply by a scaling matrix: $\mathbf{C} \leftarrow \mathbf{C\,S}$

Rotation about a Fixed Point

Start with identity matrix: $C \leftarrow I$

Move fixed point to origin: $C \leftarrow CT$

Rotate: $C \leftarrow CR$

Move fixed point back: $C \leftarrow CT^{-1}$

Result: $C = TR\,T^{-1}$ ***which is** backwards.*

This result is a consequence of doing postmultiplications.

Let's try again.

Reversing the Order

We want $C = T^{-1} R T$

so we must do the operations in the following order

$C \leftarrow I$

$C \leftarrow CT^{-1}$

$C \leftarrow CR$

$C \leftarrow CT$

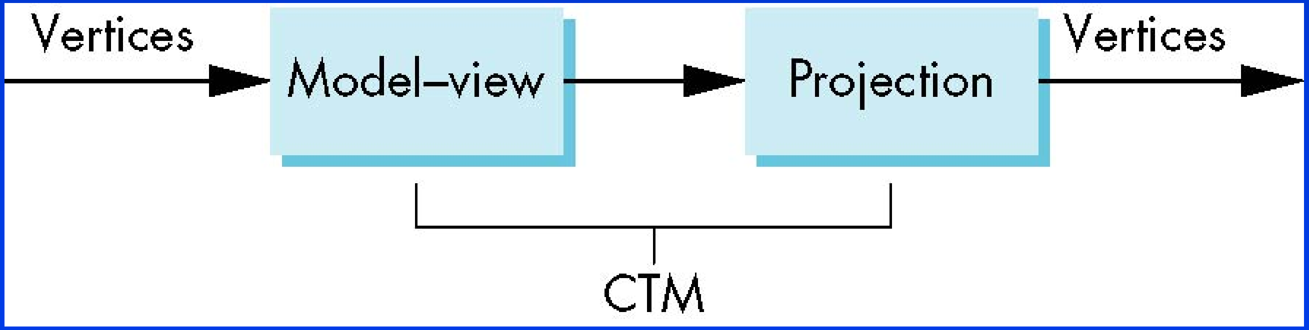
Each operation corresponds to one function call in the program.

Note that the last operation specified is the first executed in the program

CTM in OpenGL

OpenGL has a model-view and a projection matrix in the pipeline which are concatenated together to form the CTM

Can manipulate each by first setting the correct matrix mode



Rotation, Translation, Scaling

Load an identity matrix:

```
glLoadIdentity()
```

Multiply on right:

`glRotatef(theta, vx, vy, vz)`

`theta` in degrees, (**`vx, vy, vz`**) define axis of rotation

```
glTranslatef(dx, dy, dz)

glScalef( sx, sy, sz)
```

Each has a float (f) and double (d) format (**`glScaled`**)

Example

Rotation about z axis by 30 degrees with a fixed point of (1.0, 2.0, 3.0)

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
glTranslatef(1.0, 2.0, 3.0);
glRotatef(30.0, 0.0, 0.0, 1.0);
glTranslatef(-1.0, -2.0, -3.0);
```

Remember that last matrix specified in the program is the first applied

Arbitrary Matrices

Can load and multiply by matrices defined in the application program

```
glLoadMatrixf(m)
glMultMatrixf(m)
```

The matrix `m` is a one dimension array of 16 elements which are the components of the desired 4 x 4 matrix stored by <u>columns</u>

In `glMultMatrixf`, `m` multiplies the existing matrix on the right

Transformations in OpenGL

OpenGL makes it easy to do transformations to the CS, not the object

Sequence of operations:

- Set up a routine to draw the object in its “base” CS
- Call transformation routines to transform the CS
- Object drawn in transformed CS

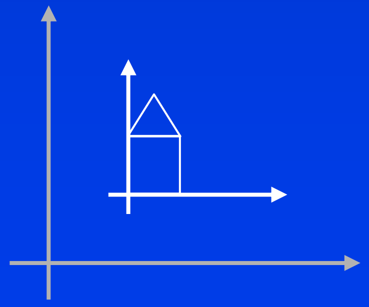
OpenGL transformation example

```
drawHouse(){
        glBegin(GL_LINE_LOOP);
        glVertex2i(0,0);
        glVertex2i(0,2);
                ….
        glEnd();
}
```

```
drawTransformedHouse(){
        glMatrixMode(GL_MODELVIEW);
        glTranslatef(4.0, 4.0, 0.0);
        glScalef(0.5, 0.5, 1.0);
        drawHouse();
}
```

Draws basic house

Draws transformed house



Matrix Stacks

In many situations we want to save transformation matrices for use later

- Traversing hierarchical data structures
- Avoiding state changes when executing display lists

OpenGL maintains stacks for each type of matrix

- Access present type (as set by **`glMatrixMode)`** by

```
glPushMatrix()
glPopMatrix()
```

OpenGL matrix stack example

```
glLoadMatrixf(m0);
glPushMatrix();
glMultMatrixf(m1);
glPushMatrix();
glMultMatrixf(m4);
render chair2;
glPopMatrix();
glPushMatrix();
glMultMatrixf(m3);
render chair1;
glPopMatrix();
render table;
glPopMatrix();
glPushMatrix();
glMultMatrixf(m2);
render rug;
glPopMatrix()
render room;
```