CS 4204 Computer Graphics

Curves and Surfaces

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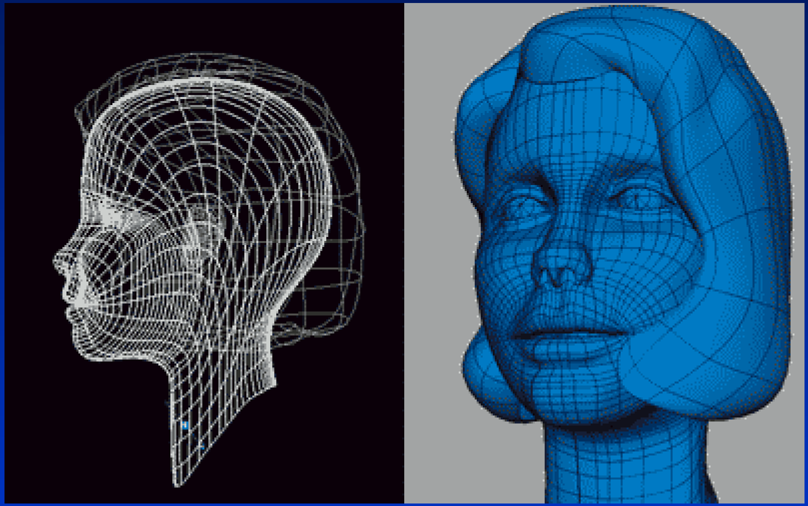
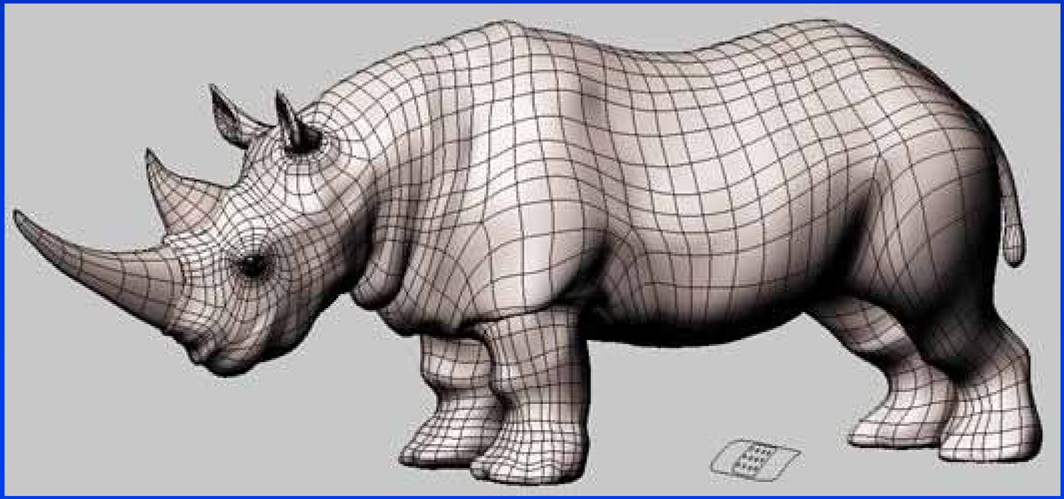
Reference: Ed Angle, Interactive Computer Graphics, University of New Mexico, class notes

Curve and Surface Modeling

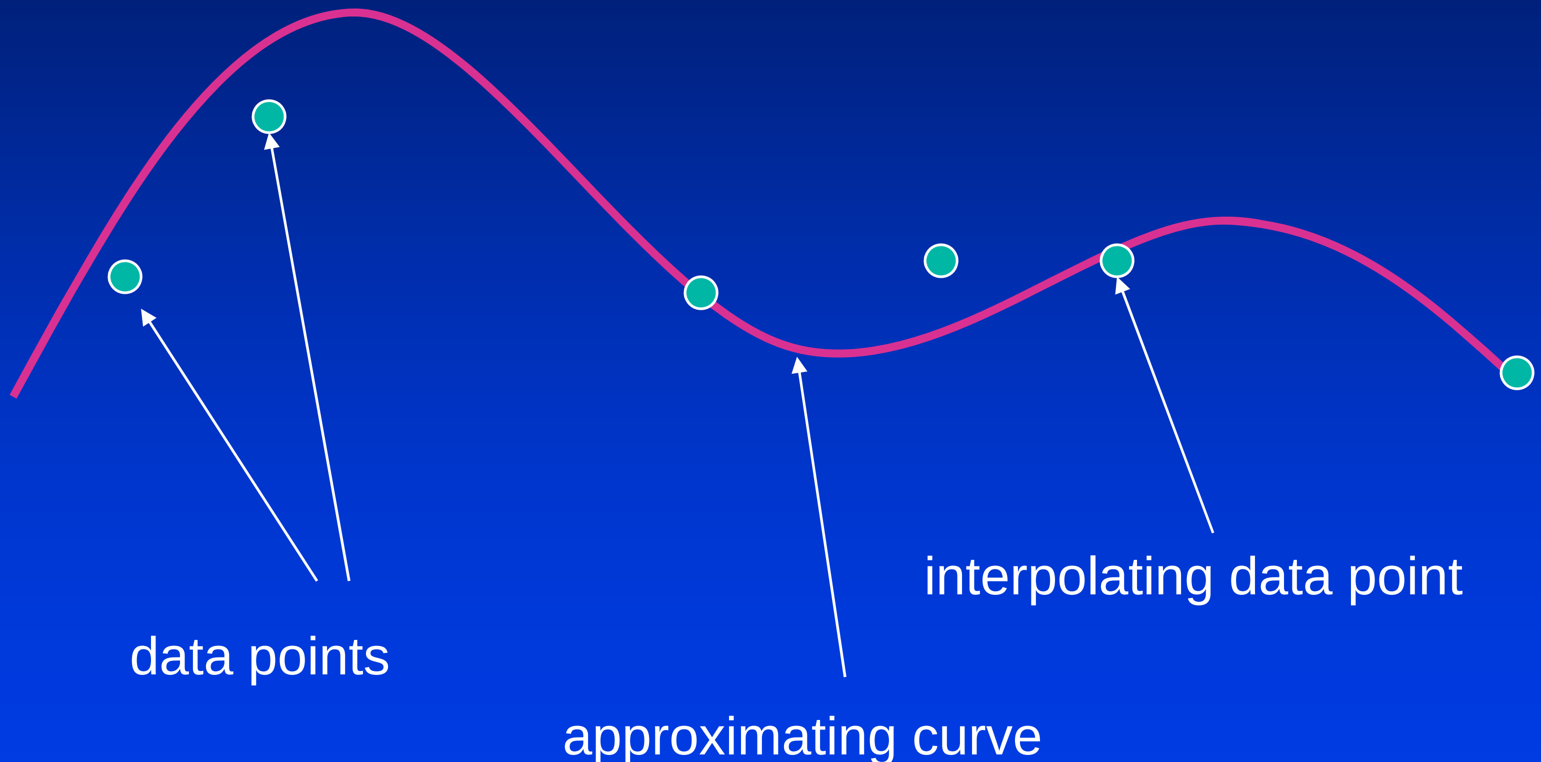




Objectives

- ***Introduce types of curves and surfaces***
  - Explicit
  - Implicit
  - Parametric
  - Strengths and weaknesses
- ***Discuss Modeling and Approximations***
  - Conditions
  - Stability

Modeling with Curves



What Makes a Good Representation?

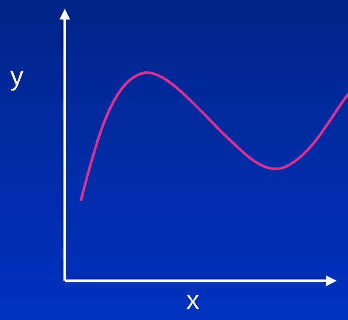
- ***There are many ways to represent curves and surfaces***
- ***Want a representation that is***
  - Stable
  - Smooth
  - Easy to evaluate

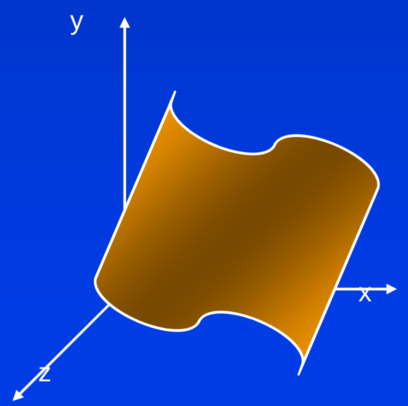
Explicit Representation

- *Most familiar form of curve in 2D*

$$y=f(x)$$

- *Cannot represent all curves*
  - *Vertical lines*
  - *Circles*
- *Extension to 3D*
  - *y=f(x), z=g(x)*
  - *The form z = f(x,y) defines a surface*





Implicit Representation

- ***Two dimensional curve(s)***

  $$g(x,y)=0$$

- ***Much more robust***
  - All lines $ax+by+c=0$
  - Circles $x^2+y^2-r^2=0$
- ***Three dimensions*** $g(x,y,z)=0$ ***defines a surface***
  - Intersect two surface to get a curve

Parametric Curves

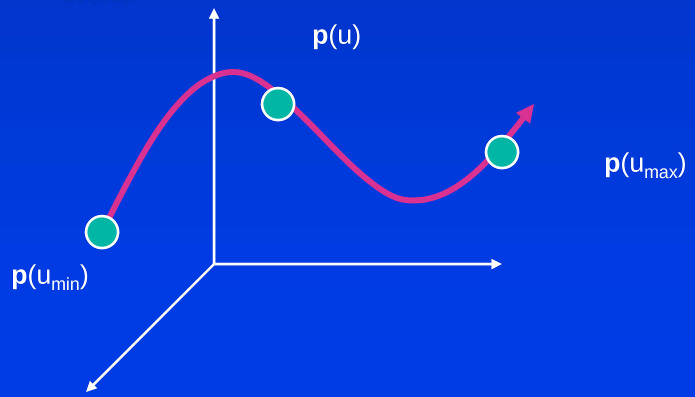
- ***Separate equation for each spatial variable***

$$x=x(u)$$

$$y=y(u) \qquad\qquad \mathbf{p}(u)=[x(u),\, y(u),\, z(u)]^T$$

$$z=z(u)$$

- ***For $u_{max} \geq u \geq u_{min}$ we trace out a curve in two or three dimensions***



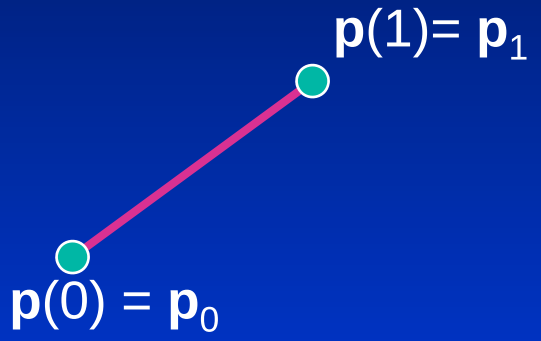
Selecting Functions

- ***Usually we can select "good" functions***
- not unique for a given spatial curve
- Approximate or interpolate known data
- Want functions which are easy to evaluate
- Want functions which are easy to differentiate
  - *Computation of normals*
  - *Connecting pieces (segments)*
- Want functions which are smooth

Parametric Lines

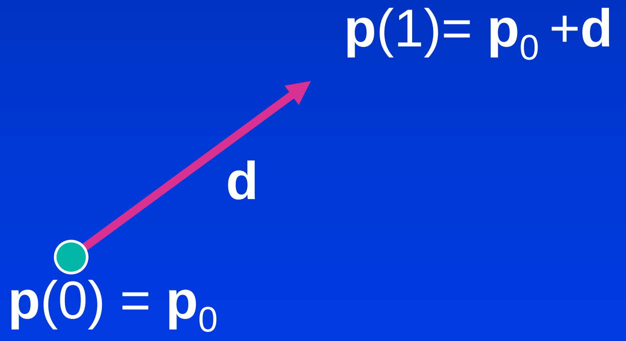
We can normalize u to be over the interval (0,1)

Line connecting two points $\mathbf{p}_0$ and $\mathbf{p}_1$



$$\mathbf{p}(u)=(1-u)\mathbf{p}_0+u\mathbf{p}_1$$

Ray from $\mathbf{p}_0$ in the direction $\mathbf{d}$



$$\mathbf{p}(u)=\mathbf{p}_0+u\mathbf{d}$$

Parametric Surfaces

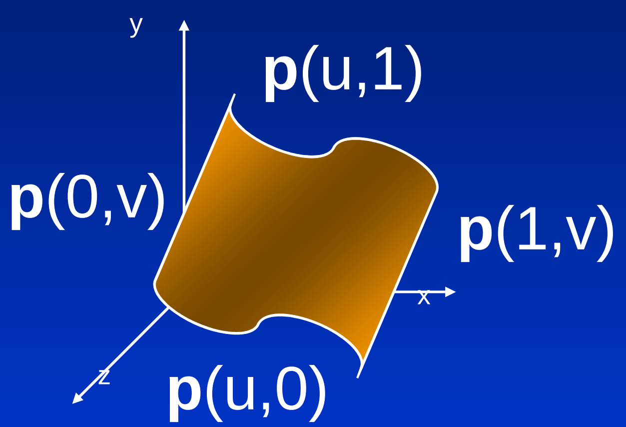
- ***Surfaces require 2 parameters***

$$x=x(u,v)$$

$$y=y(u,v)$$

$$z=z(u,v)$$

$$p(u,v) = [x(u,v), y(u,v), z(u,v)]^T$$



- ***Want same properties as curves:***
  - Smoothness
  - Differentiability
  - Ease of evaluation

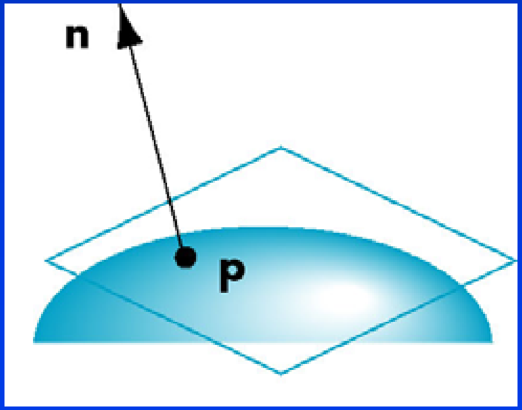
Normals

We can differentiate with respect to u and v to obtain the normal at any point $\mathbf{p}$

$$\frac{\partial \mathbf{p}(u,v)}{\partial u} = \begin{bmatrix} \partial x(u,v)/\partial u \\ \partial y(u,v)/\partial u \\ \partial z(u,v)/\partial u \end{bmatrix}$$

$$\frac{\partial \mathbf{p}(u,v)}{\partial v} = \begin{bmatrix} \partial x(u,v)/\partial v \\ \partial y(u,v)/\partial v \\ \partial z(u,v)/\partial v \end{bmatrix}$$

$$\mathbf{n} = \frac{\partial \mathbf{p}(u,v)}{\partial u} \times \frac{\partial \mathbf{p}(u,v)}{\partial v}$$



Parametric Planes

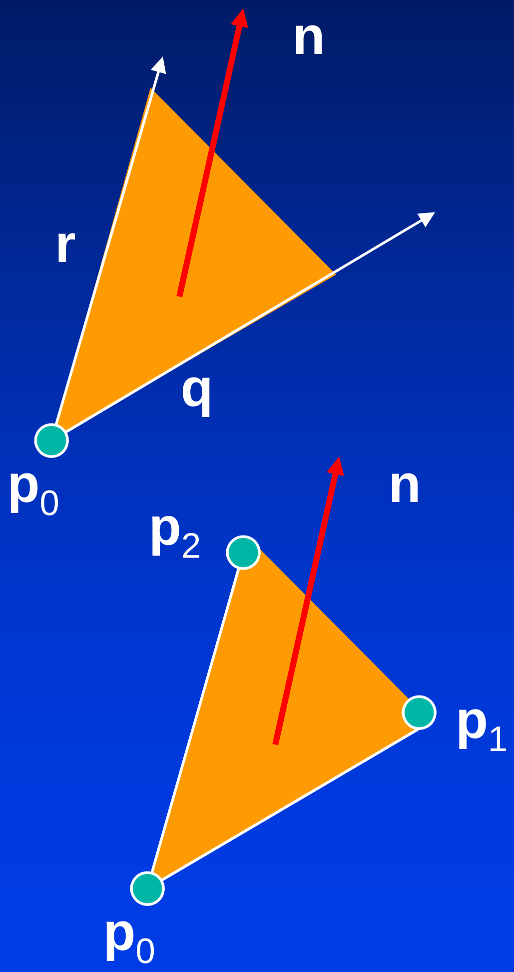
point-vector form

$\mathbf{p}(u,v)=\mathbf{p}_0+u\mathbf{q}+v\mathbf{r}$

$\mathbf{n} = \mathbf{q} \times \mathbf{r}$

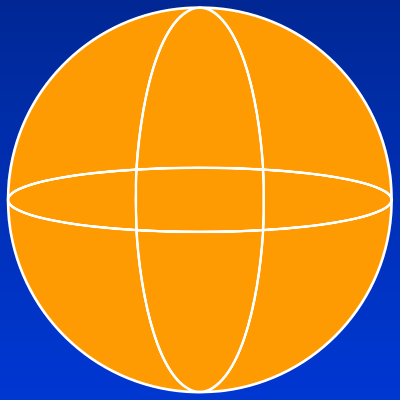
three-point form

$\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_0$
$\mathbf{r} = \mathbf{p}_2 - \mathbf{p}_0$



Parametric Sphere

x(u,v) = r cos q sin f
y(u,v) = r sin q sin f
z(u,v) = r cos f



360 ≥ q ≥ 0
180 ≥ f ≥ 0

θ constant: circles of constant longitude
f constant: circles of constant latitude

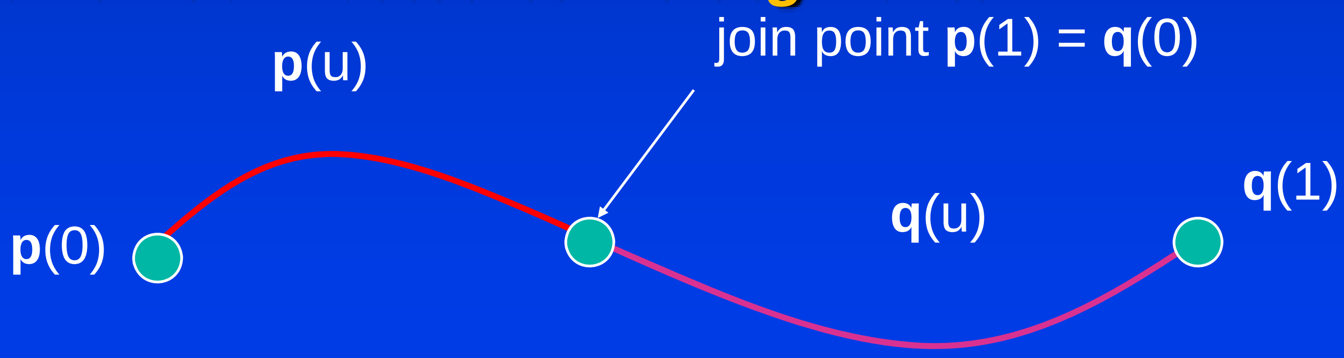
Normal: **n = p**

Curve Segments

- ***After normalizing u, each curve is written***

  $\boldsymbol{p(u)=[x(u), y(u), z(u)]^T, \quad 1 \geq u \geq 0}$

- ***In classical numerical methods, we design a single global curve***
- ***In computer graphics and CAD, it is better to design small connected curve segments***



Parametric Polynomial Curves

$$x(u) = \sum_{i=0}^{N} c_{xi} u^i \qquad y(u) = \sum_{j=0}^{M} c_{yj} u^j \qquad z(u) = \sum_{k=0}^{L} c_{zk} u^k$$

•If N=M=K, we need to determine 3(N+1) coefficients

•Equivalently we need 3(N+1) independent conditions

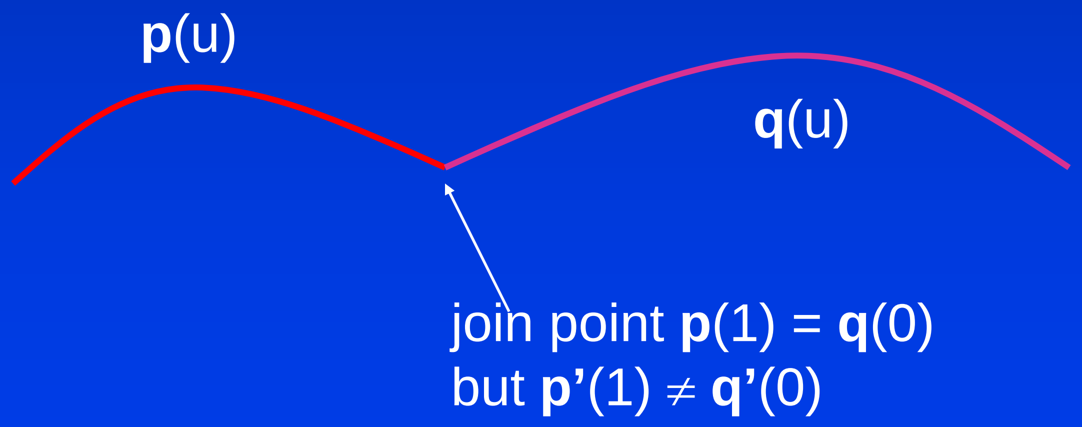
•Noting that the curves for x, y and z are independent, we can define each independently in an identical manner

•We will use the form where p can be any of x, y, z

$$\mathrm{p}(u) = \sum_{k=0}^{L} c_k u^k$$

Why Polynomials

- ***Easy to evaluate***
- ***Continuous and differentiable everywhere***
  - Must worry about continuity at join points including continuity of derivatives



Cubic Parametric Polynomials

- ***N=M=L=3, gives balance between ease of evaluation and flexibility in design***

$$p(u) = \sum_{k=0}^{3} c_k u^k$$

- ***Four coefficients to determine for each of x, y and z***
- ***Seek four independent conditions for various values of u resulting in 4 equations in 4 unknowns for each of x, y and z***
  - Conditions are a mixture of continuity requirements at the join points and conditions for fitting the data

Cubic Polynomial Surfaces

p(u,v)=[x(u,v), y(u,v), z(u,v)]^T

where

$$p(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} c_{ij} u^i v^j$$

p is any of x, y or z

Need 48 coefficients (3 independent sets of 16) to determine a surface patch

Objectives

- ***Introduce the types of curves***
  - Interpolating
  - Hermite
  - Bezier
  - B-spline
- ***Analyze their performance***

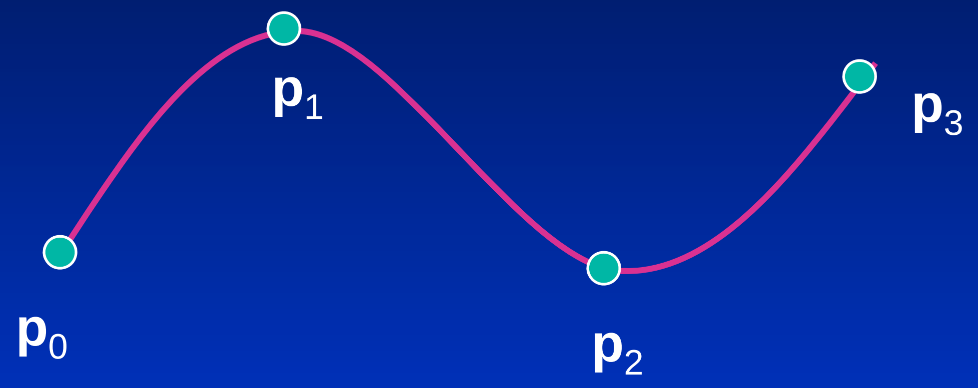
Matrix-Vector Form

$$\mathrm{p}(u) = \sum_{k=0}^{3} c_k u^k$$

define
$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}$$

then
$$\mathrm{p}(u) = \mathbf{u}^T \mathbf{c} = \mathbf{c}^T \mathbf{u}$$

Interpolating Curve



Given four data (control) points $\mathbf{p}_0$, $\mathbf{p}_1$,$\mathbf{p}_2$, $\mathbf{p}_3$
determine cubic $\mathbf{p}(u)$ which passes through them

Must find $\mathbf{c}_0$,$\mathbf{c}_1$,$\mathbf{c}_2$, $\mathbf{c}_3$

Interpolation Equations

apply the interpolating conditions at u=0, 1/3, 2/3, 1

$p_0=p(0)=c_0$
$p_1=p(1/3)=c_0+(1/3)c_1+(1/3)^2c_2+(1/3)^3c_3$
$p_2=p(2/3)=c_0+(2/3)c_1+(2/3)^2c_2+(2/3)^3c_3$
$p_3=p(1)=c_0+c_1+c_2+c_3$

or in matrix form with **p** = $[p_0\ p_1\ p_2\ p_3]^T$

p=Ac

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \left(\frac{2}{3}\right) & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Interpolation Matrix

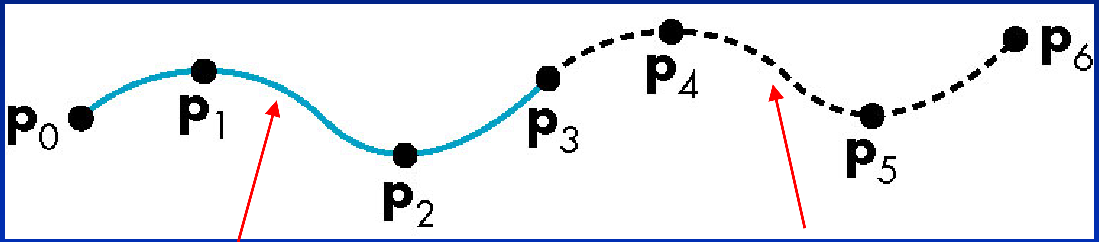
Solving for **c** we find the *interpolation matrix*

$$\mathbf{M}_I = \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & -22.5 & 18 & -4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{bmatrix}$$

$$\mathbf{c} = \mathbf{M}_I \mathbf{p}$$

Note that $\mathbf{M}_I$ does not depend on input data and can be used for each segment in x, y, and z

Interpolating Multiple Segments



use $\mathbf{p} = [p_0\ p_1\ p_2\ p_3]^T$

use $\mathbf{p} = [p_3\ p_4\ p_5\ p_6]^T$

Get continuity at join points but not continuity of derivatives

Blending Functions

Rewriting the equation for p(u)

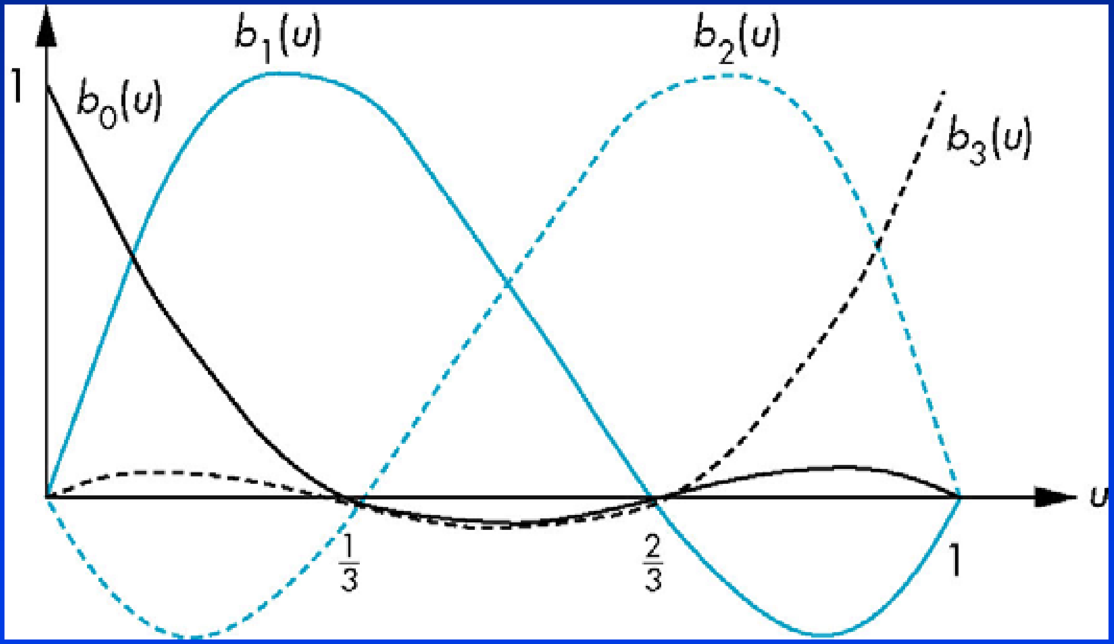
$$p(u)=\mathbf{u}^T\mathbf{c}=\mathbf{u}^T\mathbf{M}_I\mathbf{p} = \mathbf{b}(u)^T\mathbf{p}$$

where $b(u) = [b_0(u)\ b_1(u)\ b_2(u)\ b_3(u)]^T$ is an array of ***blending polynomials*** such that
$p(u) = b_0(u)p_0 + b_1(u)p_1 + b_2(u)p_2 + b_3(u)p_3$

$b_0(u) = -4.5(u-1/3)(u-2/3)(u-1)$
$b_1(u) = 13.5u\ (u-2/3)(u-1)$
$b_2(u) = -13.5u\ (u-1/3)(u-1)$
$b_3(u) = 4.5u\ (u-1/3)(u-2/3)$

Blending Functions

- ***These functions are not smooth***
  - Hence the interpolation polynomial is not smooth

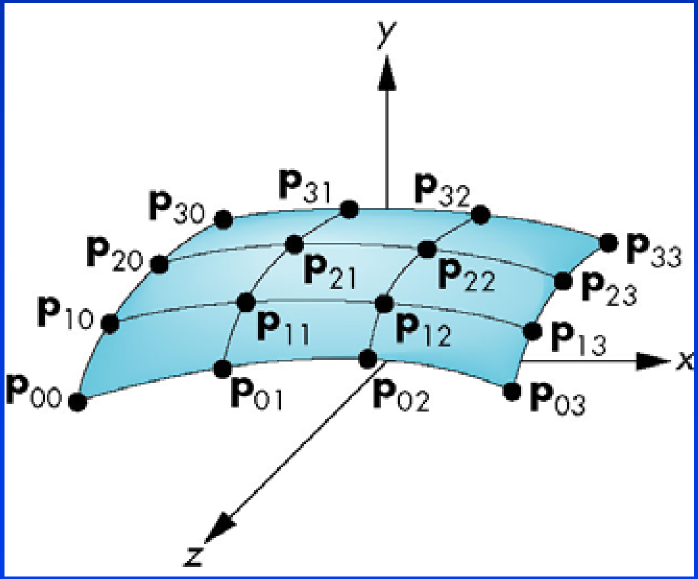


Interpolating Patch

$$p(u,v) = \sum_{i=o}^{3} \sum_{j=0}^{3} c_{ij} u^i v^j$$

Need 16 conditions to determine the 16 coefficients c_{ij}

Choose at u,v = 0, 1/3, 2/3, 1



Matrix Form

Define $\mathbf{v} = [1\ v\ v^2\ v^3]^T$

$\mathbf{C} = [c_{ij}] \qquad \mathbf{P} = [p_{ij}]$

$p(u,v) = \mathbf{u}^T C \mathbf{v}$

If we observe that for constant u (v), we obtain interpolating curve in v (u), we can show

$$\mathbf{C} = M_I P M_I$$

$$p(u,v) = u^T M_I P M_I^T v$$

Blending Patches

$$p(u,v) = \sum_{i=o}^{3} \sum_{j=0}^{3} b_i(u) b_j(v) p_{ij}$$

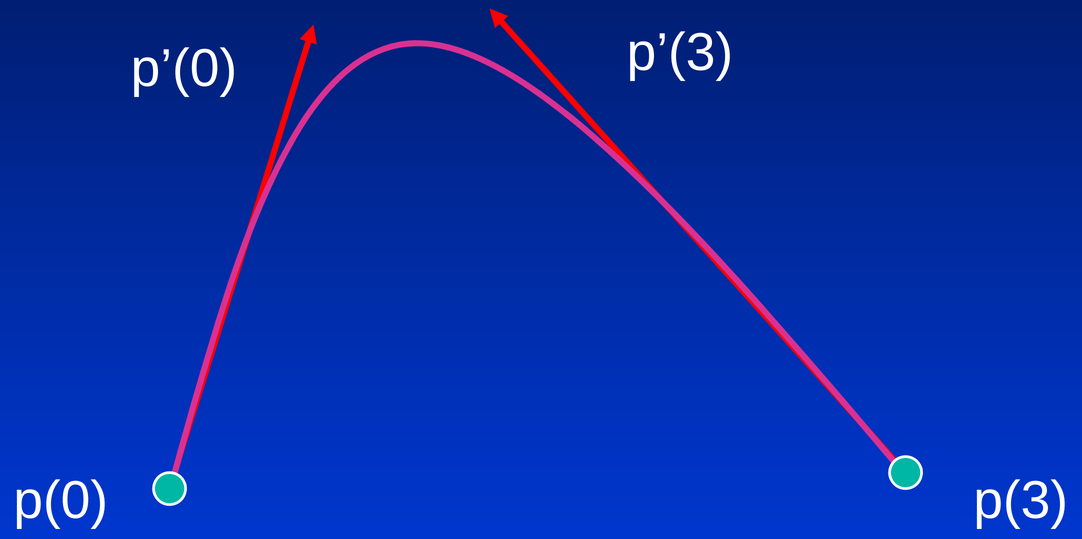
Each $b_i(u)b_j(v)$ is a blending patch

Shows that we can build and analyze surfaces from our knowledge of curves

Other Types of Curves and Surfaces

- ***How can we get around the limitations of the interpolating form***
  - Lack of smoothness
  - Discontinuous derivatives at join points
- ***We have four conditions (for cubics) that we can apply to each segment***
  - Use them other than for interpolation
  - Need only come close to the data

Hermite Form



Use two interpolating conditions and two derivative conditions per segment

Ensures continuity and first derivative continuity between segments

Equations

Interpolating conditions are the same at ends

$$p(0) = p_0 = c_0$$
$$p(1) = p_3 = c_0+c_1+c_2+c_3$$

Differentiating we find $p'(u) = c_1+2uc_2+3u^2c_3$

Evaluating at end points

$$p'(0) = p'_0 = c_1$$
$$p'(1) = p'_3 = c_1+2c_2+3c_3$$

Matrix Form

$$\mathbf{q} = \begin{bmatrix} p_0 \\ p_3 \\ p'_0 \\ p'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \mathbf{c}$$

Solving, we find $\mathbf{c}=\mathbf{M}_H\mathbf{q}$ where $\mathbf{M}_H$ is the Hermite matrix

$$\mathbf{M}_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

Blending Polynomials

$$p(u) = \mathbf{b}(u)^T\mathbf{q}$$

$$\mathbf{b}(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}$$

Although these functions are smooth, the Hermite form is not used directly in Computer Graphics and CAD because we usually have control points but not derivatives

However, the Hermite form is the basis of the Bezier form

Parametric and Geometric Continuity

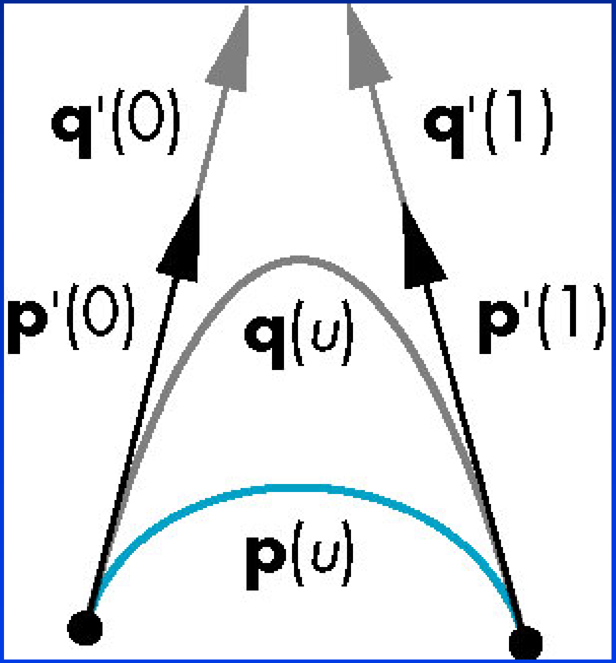
- ***We can require the derivatives of x, y,and z to each be continuous at join points (parametric continuity)***
- ***Alternately, we can only require that the tangents of the resulting curve be continuous (geometry continuity)***
- ***The latter gives more flexibility as we have need satisfy only two conditions rather than three at each join point***

Example

- *Here the p and q have the same tangents at the ends of the segment but different derivatives*
- *Generate different*

  Hermite curves
- *This techniques is used*

in drawing applications



Higher Dimensional Approximations

- ***The techniques for both interpolating and Hermite curves can be used with higher dimensional parametric polynomials***
- ***For interpolating form, the resulting matrix becomes increasingly more ill-conditioned and the resulting curves less smooth and more prone to numerical errors***
- ***In both cases, there is more work in rendering the resulting polynomial curves and surfaces***