Chapter 9: Schemes for the One-Dimensional Nonlinear Shallow-Water Equations

9.1 Properties of the continuous equations

In this chapter, we consider a highly idealized version of the momentum equation: Shallow water, one dimension, no rotation. In a later chapter, we will go to two dimensions with rotation, and bring in the effects of vorticity, which are extremely important.

Consider the one-dimensional shallow-water equations, with bottom topography, without rotation and with $v = 0$. The prognostic variables are the water depth or mass, h, and the speed, u. The exact equations are

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 , \tag{1}$$

and

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[K + g\left(h + h_S\right)\right] = 0 . \tag{2}$$

Here

$$K \equiv \frac{1}{2}u^2 \tag{3}$$

is the kinetic energy per unit mass, g is the acceleration of gravity, and h_S is the height of the bottom topography. In Eq. (2), the vorticity has been assumed to vanish, which is reasonable in the absence of rotation and in one dimension. The effects of vorticity are of course absolutely critical in geophysical fluid dynamics; they will be discussed in a later chapter.

The design of the scheme is determined by a sequence of choices. We should welcome the opportunity to make the *best possible* choices. The first thing that we have to choose is the particular form of the continuous equations that the space-differencing scheme is designed to mimic. Eq. (2) is one possible choice for the continuous form of the momentum equation. An alternative choice is

$$\frac{\partial}{\partial t}(hu)+\frac{\partial}{\partial x}(huu)+gh\frac{\partial}{\partial x}(h+h_S)=0\,, \tag{4}$$

i.e., the flux form of the momentum equation, which can be derived by combining (1) and (2).

The continuous shallow-water equations have important "integral properties," which we will use as a guide in the design of our space-differencing scheme. For example, if we integrate (1) with respect to x, over a closed or periodic domain, we obtain

$$\frac{d}{dt}\left(\int_{\text{domain}} h\,dx\right)=0\,, \tag{5}$$

which means that mass is conserved.

Using

$$h\frac{\partial h}{\partial x}=\frac{\partial}{\partial x}\left(\frac{h^2}{2}\right), \tag{6}$$

we can rewrite (4) as

$$\frac{\partial}{\partial t}(hu)+\frac{\partial}{\partial x}\left(huu+g\frac{h^2}{2}\right)=-gh\frac{\partial h_S}{\partial x}\,. \tag{7}$$

The momentum per unit area is hu. If we integrate with respect to x, over a periodic domain, we obtain

$$\frac{d}{dt}\left(\int_{\text{domain}} hu\,dx\right)=-\int_{\text{domain}} gh\frac{\partial h_S}{\partial x}dx\,. \tag{8}$$

This shows that in the absence of topography, i.e., if $\frac{\partial h_S}{\partial x}=0$ everywhere, the domain average of hu is invariant, i.e., momentum is conserved. When h_S is spatially variable, the atmosphere and the "solid earth" can exchange momentum through the pressure field.

The flux form of the kinetic energy equation can be derived by multiplying (1) by K and (2) by hu, and adding the results, to obtain

$$\frac{\partial}{\partial t}(hK)+\frac{\partial}{\partial x}(huK)+hu\frac{\partial}{\partial x}\left[g\left(h+h_S\right)\right]=0\,. \tag{9}$$

The last term of (9) represents conversion between potential and kinetic energy.

The potential energy equation can be derived by multiplying (1) by $g\left(h+h_S\right)$ to obtain

$$\frac{\partial}{\partial t}\left[hg\left(h_S+\frac{1}{2}h\right)\right]+g\left(h+h_S\right)\frac{\partial}{\partial x}(hu)=0\,, \tag{10}$$

or

$$\frac{\partial}{\partial t}\left[hg\left(h_S+\frac{1}{2}h\right)\right]+\frac{\partial}{\partial x}\left[hug\left(h+h_S\right)\right]-hu\frac{\partial}{\partial x}\left[g\left(h+h_S\right)\right]=0\,. \tag{11}$$

The last term of (11) represents conversion between kinetic and potential energy; compare with (9). In deriving (10), we have assumed that h_S is independent of time. This assumption can easily be relaxed.

When we add (9) and (10), the energy conversion terms cancel, and we obtain a statement of the conservation of total energy, i.e.,

$$\frac{\partial}{\partial t}\left\{h\left[K+g\left(h_S+\frac{1}{2}h\right)\right]\right\}+\frac{\partial}{\partial x}\left\{hu\left[K+g\left(h+h_S\right)\right]\right\}=0\,. \tag{12}$$

The integral of (10) over a closed or periodic domain gives

$$\frac{d}{dt}\int_{\text{domain}} h\left[K+g\left(h_S+\frac{1}{2}h\right)\right]dx=0\,, \tag{13}$$

which shows that the domain-integrated total energy is conserved.

9.2 Space differencing

Now consider finite-difference approximations to (1) and (2). We keep the time derivatives continuous, and explore the effects of space differencing only. *We use a staggered grid, with h defined at integer points (hereafter called mass points) and u at half-integer points (hereafter called wind points).* This can be viewed as a one-dimensional version of the C grid. The grid spacing, Δx, is assumed to be uniform. .Our selection of this particular grid is a second *choice* made in the design of the space-differencing scheme.

The finite difference version of the mass conservation equation is

$$\frac{dh_i}{dt}+\left[\frac{(hu)_{i+\frac{1}{2}}-(hu)_{i-\frac{1}{2}}}{\Delta x}\right]=0\ .$$

(14)

It should be understood that

$$h_{i+\frac{1}{2}}u_{i+\frac{1}{2}}\equiv(hu)_{i+\frac{1}{2}}\ .$$

(15)

The "wind-point" masses, e.g., $h_{i+\frac{1}{2}}$, are undefined at this stage. The finite-difference approximation used in (14) is consistent with second-order accuracy in space, although we cannot really determine the order of accuracy until the finite-difference form of the mass flux has been specified. We have already discussed how the "flux form" of (14) makes it possible for the model to conserve mass, regardless of how the mass fluxes are defined, i.e.,

$$\frac{d}{dt}\left(\sum_{\text{domain}}h_i\right)=0\ .$$

(16)

This is analogous to (5).

The finite-difference momentum equation that is modeled after (2) is

$$\frac{du_{i+\frac{1}{2}}}{dt}+\left(\frac{K_{i+1}-K_i}{\Delta x}\right)+g\left[\frac{(h+h_S)_{i+1}-(h+h_S)_i}{\Delta x}\right]=0\ .$$

(17)

The kinetic energy per unit mass, K_i , is undefined at this stage, but resides at mass points. The finite-difference approximations used in (17) are consistent with second-order accuracy in space, although we cannot really determine the order of accuracy until the finite-difference forms of the mass flux and kinetic energy are specified. Multiply (17) by $h_{i+\frac{1}{2}}$ to obtain

$$h_{i+\frac{1}{2}}\frac{du_{i+\frac{1}{2}}}{dt}+h_{i+\frac{1}{2}}\left(\frac{K_{i+1}-K_i}{\Delta x}\right)+gh_{i+\frac{1}{2}}\left[\frac{(h+h_S)_{i+1}-(h+h_S)_i}{\Delta x}\right]=0\ .$$

(18)

In order to mimic the differential relationship (6), we must require that

$$h_{i+\frac{1}{2}}\left(\frac{h_{i+1}-h_i}{\Delta x}\right)=\left(\frac{h_{i+1}^2-h_i^2}{2\Delta x}\right),$$

(19)

which leads to

$$h_{i+\frac{1}{2}} = \frac{h_{i+1} + h_i}{2} . \tag{20}$$

This choice is required for momentum conservation. In view of (20), we can write

$$(hu)_{i+\frac{1}{2}} = \left(\frac{h_{i+1} + h_i}{2} \right) u_{i+\frac{1}{2}} . \tag{21}$$

Combining (20) with the continuity equation (14), we see that we can write a *continuity equation for the wind points*, as follows:

$$\frac{dh_{i+\frac{1}{2}}}{dt} + \frac{1}{2\Delta x}\left[(hu)_{i+\frac{3}{2}} - (hu)_{i-\frac{1}{2}} \right] = 0 . \tag{22}$$

It should be clear from the form of (22) that the "wind-point mass" is actually conserved by the model. Of course, we do not actually use (22) when we integrate the model; instead we use (14). Nevertheless, (22) will be satisfied, because it can be derived from (14) and (20). An alternative form of (22) is

$$\frac{dh_{i+\frac{1}{2}}}{dt} + \frac{1}{\Delta x}\left[(hu)_{i+1} - (hu)_i \right] = 0 , \tag{23}$$

where

$$(hu)_{i+1} \equiv \frac{1}{2}\left[(hu)_{i+\frac{3}{2}} + (hu)_{i+\frac{1}{2}} \right] \text{ and } (hu)_i \equiv \frac{1}{2}\left[(hu)_{i+\frac{1}{2}} + (hu)_{i-\frac{1}{2}} \right]. \tag{24}$$

Now add (18) and $u_{i+\frac{1}{2}}$ times (23), and use (19), to obtain what "should be" the flux form of the momentum equation, analogous to (4):

$$\frac{d}{dt}\left(h_{i+\frac{1}{2}} u_{i+\frac{1}{2}} \right) + h_{i+\frac{1}{2}} \left(\frac{K_{i+1} - K_i}{\Delta x} \right) + \frac{u_{i+\frac{1}{2}}\left[(hu)_{i+1} - (hu)_i \right]}{\Delta x} + g\left(\frac{h_{i+1}^2 - h_i^2}{2\Delta x} \right) = -gh_{i+\frac{1}{2}}\left[\frac{(h_S)_{i+1} - (h_S)_i}{\Delta x} \right]. \tag{25}$$

Suppose that the kinetic energy is defined by

$$K_i \equiv \frac{1}{2} u_{i+\frac{1}{2}} u_{i-\frac{1}{2}} .$$

(26)

Other possible definitions of K_i will be discussed later. Using (26) and (24), we can write

$$\begin{aligned} & h_{i+\frac{1}{2}} \left(\frac{K_{i+1} - K_i}{\Delta x} \right) + u_{i+\frac{1}{2}} \frac{1}{\Delta x} \left[(hu)_{i+1} - (hu)_i \right] \\ &= \frac{1}{2\Delta x} \left\{ h_{i+\frac{1}{2}} \left(u_{i+\frac{3}{2}} u_{i+\frac{1}{2}} - u_{i+\frac{1}{2}} u_{i-\frac{1}{2}} \right) + u_{i+\frac{1}{2}} \left[(hu)_{i+\frac{3}{2}} - (hu)_{i-\frac{1}{2}} \right] \right\} \\ &= \frac{1}{\Delta x} \left[\left(\frac{h_{i+\frac{1}{2}} + h_{i+\frac{3}{2}}}{2} \right) u_{i+\frac{3}{2}} u_{i+\frac{1}{2}} - \left(\frac{h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}}}{2} \right) u_{i-\frac{1}{2}} u_{i+\frac{1}{2}} \right] . \end{aligned}$$

(27)

This is a flux form. The momentum flux at the point i is $\left(\frac{h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}}}{2} \right) u_{i-\frac{1}{2}} u_{i+\frac{1}{2}}$, and the momentum flux at the point $i+1$ is $\left(\frac{h_{i+\frac{1}{2}} + h_{i+\frac{3}{2}}}{2} \right) u_{i+\frac{3}{2}} u_{i+\frac{1}{2}}$. Because (27) is a flux form, momentum will be conserved by the scheme *if we define the kinetic energy by* (26).

Note, however, that there are two problems with (26) when when u is dominated by the $2\Delta x$-mode. For starters, (26) will give a negative value of K_i, which is unphysical. As a result, the momentum flux will be negative for the $2\Delta x$-mode, i.e., momentum will be transferred in the $-x$ direction, assuming that the interpolated masses that appear in the momentum fluxes are positive.

Next, we derive the kinetic energy equation. Recall that the kinetic energy is defined at mass points. To begin the derivation, multiply (17) by $(hu)_{i+\frac{1}{2}}$ to obtain

$$(hu)_{i+\frac{1}{2}} \frac{du_{i+\frac{1}{2}}}{dt} + (hu)_{i+\frac{1}{2}} \left(\frac{K_{i+1} - K_i}{\Delta x} \right) + g (hu)_{i+\frac{1}{2}} \left[\frac{(h+h_S)_{i+1} - (h+h_S)_i}{\Delta x} \right] = 0 .$$

(28)

Here we have returned to a general form of K_i; Eq. (26) is not being used. Rewrite (28) for grid point $i - \frac{1}{2}$, simply by subtracting one from each subscript:

$$(hu)_{i-\frac{1}{2}}\frac{du_{i-\frac{1}{2}}}{dt}+(hu)_{i-\frac{1}{2}}\left(\frac{K_{i+1}-K_i}{\Delta x}\right)+g(hu)_{i-\frac{1}{2}}\left[\frac{(h+h_S)_i-(h+h_S)_{i-1}}{\Delta x}\right]=0\ . \tag{29}$$

Now add (28) and (29), and multiply the result by $\frac{1}{2}$:

$$\begin{aligned}&\frac{1}{2}\left[(hu)_{i+\frac{1}{2}}\frac{du_{i+\frac{1}{2}}}{dt}+(hu)_{i-\frac{1}{2}}\frac{du_{i-\frac{1}{2}}}{dt}\right]\\&+\frac{1}{2}\left[(hu)_{i+\frac{1}{2}}\left(\frac{K_{i+1}-K_i}{\Delta x}\right)+(hu)_{i-\frac{1}{2}}\left(\frac{K_i-K_{i-1}}{\Delta x}\right)\right]\\&+\frac{g}{2}\left\{(hu)_{i+\frac{1}{2}}\left[\frac{(h+h_S)_{i+1}-(h+h_S)_i}{\Delta x}\right]+(hu)_{i-\frac{1}{2}}\left[\frac{(h+h_S)_i-(h+h_S)_{i-1}}{\Delta x}\right]\right\}=0\ .\end{aligned} \tag{30}$$

This is an advective form of the kinetic energy equation.

Now we try to derive, from (30) and (14), a flux form of the kinetic energy equation. Begin by multiplying (14) by K_i:

$$K_i\left\{\frac{dh_i}{dt}+\left[\frac{(hu)_{i+\frac{1}{2}}-(hu)_{i-\frac{1}{2}}}{\Delta x}\right]\right\}=0\ . \tag{31}$$

Keep in mind that we still do not know what K_i is; we have just multiplied the continuity equation by a mystery variable. Add (31) and (30) to obtain

$$\begin{aligned}&K_i\frac{dh_i}{dt}+\frac{1}{2}\left[(hu)_{i+\frac{1}{2}}\frac{du_{i+\frac{1}{2}}}{dt}+(hu)_{i-\frac{1}{2}}\frac{du_{i-\frac{1}{2}}}{dt}\right]\\&+\left\{\frac{(hu)_{i+\frac{1}{2}}}{\Delta x}\left[K_i+\frac{1}{2}(K_{i+1}-K_i)\right]-\frac{(hu)_{i-\frac{1}{2}}}{\Delta x}\left[K_i-\frac{1}{2}(K_i-K_{i-1})\right]\right\}\\&+g\left\{(hu)_{i+\frac{1}{2}}\frac{\left[(h+h_S)_{i+1}-(h+h_S)_i\right]}{2\Delta x}+(hu)_{i-\frac{1}{2}}\frac{\left[(h+h_S)_i-(h+h_S)_{i-1}\right]}{2\Delta x}\right\}=0\ .\end{aligned} \tag{32}$$

Eq. (32) “should” be a flux form of the kinetic energy equation.

The advection terms on the second line of (32) are very easy to deal with. They can be rearranged to

$$\frac{1}{\Delta x}\left[(hu)_{i+\frac{1}{2}}\frac{1}{2}\left(K_{i+1}+K_i\right)-(hu)_{i-\frac{1}{2}}\frac{1}{2}\left(K_i+K_{i-1}\right)\right]. \tag{33}$$

This has the form of a "finite-difference flux divergence." The conclusion is that these terms are consistent with kinetic energy conservation under advection, simply by virtue of their form, regardless of the method chosen to determine K_i.

Next, consider the energy conversion terms on the third line of (32), i.e.,

$$g\left\{(hu)_{i+\frac{1}{2}}\left[\frac{(h+h_S)_{i+1}-(h+h_S)_i}{2\Delta x}\right]+(hu)_{i-\frac{1}{2}}\left[\frac{(h+h_S)_i-(h+h_S)_{i-1}}{2\Delta x}\right]\right\}. \tag{34}$$

We want to compare these terms with the corresponding terms of the finite-difference form of the potential energy equation, which can be derived by multiplying (14) by $g(h+h_S)_i$:

$$\frac{d}{dt}\left[h_i g\left(h_S+\frac{1}{2}h\right)_i\right]+g(h+h_S)_i\left[\frac{(hu)_{i+\frac{1}{2}}-(hu)_{i-\frac{1}{2}}}{\Delta x}\right]=0\ . \tag{35}$$

Eq. (35) is analogous to (10). We want to recast (35) so that we see advection of potential energy, as well as the energy conversion term corresponding to (34); compare with (11). We write

$$\begin{aligned}&\frac{d}{dt}\left[h_i g\left(h_S+\frac{1}{2}h\right)_i\right]+ADV_i\\&-\frac{g}{2}\left\{(hu)_{i+\frac{1}{2}}\left[\frac{(h+h_S)_{i+1}-(h+h_S)_i}{\Delta x}\right]+(hu)_{i-\frac{1}{2}}\left[\frac{(h+h_S)_i-(h+h_S)_{i-1}}{\Delta x}\right]\right\}=0\ .\end{aligned} \tag{36}$$

where "ADV_i" represents the advection of potential energy, in flux form. The second line of (36) is a copy of the energy conversion terms of (32), but with the sign reversed. We require that (36) be equivalent to (35), and ask what form of ADV_i is implied by this requirement. The answer is:

$$ADV_i=\frac{1}{\Delta x}\left\{(hu)_{i+\frac{1}{2}}\frac{g}{2}\left[(h+h_S)_{i+1}+(h+h_S)_i\right]-(hu)_{i-\frac{1}{2}}\frac{g}{2}\left[(h+h_S)_i+(h+h_S)_{i-1}\right]\right\}. \tag{37}$$

This has the form of a finite-difference flux divergence, as desired. In summary, conservation of potential energy and the cancellation of the energy conversion terms have both turned out to be pretty easy.

We are not quite finished, however, because we have not yet examined the time-rate-of-change terms of (32). Obviously, the first line of (32) must be analogous to $\frac{\partial}{\partial t}(hK)$. For convenience, we define

$$(KE \text{ tendency})_i \equiv K_i \frac{dh_i}{dt} + \frac{1}{2}\left[(hu)_{i+\frac{1}{2}} \frac{d}{dt} u_{i+\frac{1}{2}} + (hu)_{i-\frac{1}{2}} \frac{d}{dt} u_{i-\frac{1}{2}}\right]. \tag{38}$$

Substituting for the mass fluxes from (21), we can write (38) as

$$(KE \text{ tendency})_i \equiv K_i \frac{dh_i}{dt} + \frac{1}{8}\left[(h_{i+1} + h_i)\frac{d}{dt}\left(u^2_{i+\frac{1}{2}}\right) + (h_i + h_{i-1})\frac{d}{dt}\left(u^2_{i-\frac{1}{2}}\right)\right]. \tag{39}$$

The requirement for kinetic energy conservation is

$$\sum_{\text{domain}} (KE \text{ tendency})_i = \sum_{\text{domain}} \frac{d}{dt}(h_i K_i) . \tag{40}$$

Note that only the sums over *i* must agree; it is not necessary that

$$K_i \frac{dh_i}{dt} + \frac{1}{8}\left[(h_{i+1} + h_i)\frac{d}{dt}\left(u^2_{i+\frac{1}{2}}\right) + (h_i + h_{i-1})\frac{d}{dt}\left(u^2_{i-\frac{1}{2}}\right)\right] \text{ be equal to } \frac{d}{dt}(h_i K_i) \text{ for each } i.$$

To complete our check of kinetic energy conservation, we substitute for K_i on the right-hand side of (40), and check to see whether the resulting equation is actually satisfied.

The bad news is that, if we use (26), Eq. (40) is not satisfied. This means that we cannot have both momentum conservation under advection and kinetic energy conservation, when we start from the continuous form of (2). On the other hand, we did not like (26) anyway.

The good news is that there are ways to satisfy (40). Two alternative definitions of the kinetic energy are

$$K_i \equiv \frac{1}{4}\left(u^2_{i+\frac{1}{2}} + u^2_{1-\frac{1}{2}}\right), \tag{41}$$

and

$$h_i K_i \equiv \frac{1}{4}\left(h_{i+\frac{1}{2}} u^2_{i+\frac{1}{2}} + h_{i-\frac{1}{2}} u^2_{i-\frac{1}{2}} \right). \tag{42}$$

With either of these definitions, K_i cannot be negative. We can show that the sum over the domain of $h_i K_i$ given by (41) is equal to the sum over the domain of $h_i K_i$ given by (42). Either choice allows (40) to be satisfied, so both are consistent with kinetic energy conservation under advection, but neither is consistent with momentum conservation under advection.

In summary, when we start from the continuous form of (2), we can have either momentum conservation under advection or kinetic energy conservation under advection, but not both. Which is better depends on the application.

An alternative approach is to start from a finite-difference form of the momentum equation that mimics (4). In that case, we can conserve *both* momentum under advection and kinetic energy under advection. You are asked to demonstrate this in Problem 2, below.

When we generalize to the two-dimensional shallow-water equations with rotation, there are very important additional considerations having to do with vorticity, and the issues discussed here have to be revisited. This is discussed in Chapter 11.

9.3 Summary

We have explored the conservation properties of spatial finite-difference approximations of the momentum and continuity equations for one-dimensional non-rotating flow, using a staggered grid. We were able to find a scheme that guarantees conservation of mass, conservation of momentum in the absence of bottom topography, conservation of kinetic energy under advection, conservation of potential energy under advection, and conservation of total energy in the presence of energy conversion terms.

This chapter has introduced several new things. This is the first time that we have considered the momentum equation. This is the first time that we have discussed energy conversions and total energy conservation. And the chapter illustrates a way of thinking about the trade-offs that must be weighed in the design of a scheme, as various alternative choices each have advantages and disadvantages.

Problems

1. Show that if we use (26) it is not possible to conserve kinetic energy under advection.
2. Starting from a finite-difference form that mimics (4), show that it is possible to conserve both momentum and total energy. Use the C grid, and keep the time derivatives continuous.