Chapter 10: Stairways to Heaven

10.1 Introduction

Vertical differencing is a very different problem from horizontal differencing, for three reasons.

- First, gravitational effects strongly control vertical motions, and gravitational potential energy is an important source of atmospheric kinetic energy.
- Second, the Earth's atmosphere is very shallow compared to its horizontal extent, so that, *on large horizontal scales*, vertical gradients are much stronger than horizontal gradients, and horizontal motions are much faster than vertical motions. The strong vertical gradients require a high-resolution vertical grid. The high-resolution vertical grid can require small time steps to maintain computational stability.
- Third, the atmosphere has a complex lower boundary that can strongly influence the circulation through both mechanical blocking and thermal forcing.

To construct a vertically discrete model, we have to make a lot of choices, including these:

- The governing equations: Quasi-static or not? Shallow atmosphere or not? Anelastic or not?
- The vertical coordinate system;
- The vertical staggering of the model's dependent variables;
- The properties of the exact equations that we want the discrete equations to mimic.

As usual, these choices will involve trade-offs. Each possible choice will have strengths and weaknesses.

We must also be aware of possible interactions between the vertical differencing and the horizontal and temporal differencing.

10.2 Choice of equation set

The speed of sound in the Earth's atmosphere is about 300 $\mathrm{m\,s^{-1}}$. If we permit vertically propagating sound waves, then, with explicit time differencing, the largest time step that is compatible with linear computational stability can be quite small. For example, if a model has a vertical grid spacing on the order of 300 m, the allowed time step will be on the order of one second. This may be palatable if the horizontal and vertical grid spacings are comparable. On the other hand, with a horizontal grid spacing of 30 km and a vertical grid spacing of 300 m, vertically propagating sound waves will limit the time step to about one percent of the value that would be compatible with the horizontal grid spacing. That's hard to take.

There are four possible ways around this problem. One approach is to use a set of equations that filters sound waves, i.e., "anelastic" equations. There are several varieties of anelastic systems, developed over a period of forty years or so (Ogura and Phillips, 1962; Lipps and Hemler, 1982; Durran, 1989; Bannon, 1996; Arakawa and Konor, 2009). The most recent formulations are quite attractive. Anelastic models are very widely used, especially for high-resolution modeling, and anelastic systems can be an excellent choice.

A second approach is to adopt the quasi-static system of equations, in which the equation of vertical motion is replaced by the hydrostatic equation. The quasi-static system filters vertically propagating sound waves, while permitting Lamb waves, which are sound waves that propagate only in the horizontal. The quasi-static approximation is widely used in global models for both weather prediction and climate, but its errors become larger on smaller spatial scales, so its use is limited to models with horizontal grid spacings on the order of 10 km or larger, depending on the particular application.

The third approach is to use implicit or partially implicit time differencing, which can permit a long time step even when vertically propagating sound waves occur. The main disadvantage is complexity.

The fourth approach is to "sub-cycle." Small time steps can be used to integrate the terms of the equations that govern sound waves, while longer time steps are used for the remaining terms.

*** Guest lecture on "sound-proof" systems, by Celal Konor.

10.3 General vertical coordinate

*** Re-do without using the hydrostatic approximation.

The most obvious choice of vertical coordinate system, and one of the least useful, is height. As you probably already know, the equations of motion are frequently expressed using vertical coordinates other than height. The most basic requirement for a variable to be used as a vertical coordinate is that it vary monotonically with height. Even this requirement can be relaxed; e.g., a vertical coordinate can be independent of height over some layer of the atmosphere, provided that the layer is not too deep.

Factors to be weighed in choosing a vertical coordinate system for a particular application include the following:

- the form of the lower boundary condition (simpler is better);
- the form of the continuity equation (simpler is better);
- the form of the horizontal pressure gradient force (simpler is better, and a pure gradient is particularly good);
- the form of the hydrostatic equation (simpler is better);
- the intensity of the “vertical motion” as seen in the coordinate system (weaker vertical motion is simpler and better);
- the method used to compute the vertical motion (simpler is better).

Each of these factors will be discussed below, for specific vertical coordinates. We begin, however, by presenting the basic governing equations, *for quasi-static motions*, using a general vertical coordinate.

Kasahara (1974) published a detailed discussion of *general* vertical coordinates for *quasi-static* models. A more modern discussion of the same subject is given by Konor and Arakawa (1997). With a general vertical coordinate, ζ, the hydrostatic equation can be expressed as

$$\begin{aligned}\frac{\partial\phi}{\partial\zeta} &= \left(\frac{\partial\phi}{\partial p}\right)\left(\frac{\partial p}{\partial\zeta}\right)\\ &= \alpha\rho_\zeta ,\end{aligned} \tag{1}$$

where $\phi \equiv gz$ is the geopotential, g is the acceleration of gravity, z is height, p is the pressure, α is the specific volume, and ρ_ζ is the “pseudo-density” for ζ. In deriving (1), we have used the hydrostatic equation in the form

$$\frac{\partial\phi}{\partial p} = -\alpha , \tag{2}$$

and we define

$$\rho_\zeta \equiv -\left(\frac{\partial p}{\partial\zeta}\right) \tag{3}$$

as the pseudo-density, i.e., the amount of mass (as measured by the pressure difference) between two ζ-surfaces. The minus sign in (3) is arbitrary, and can be included or not according to taste, perhaps depending on the particular choice of ζ. It is also possible to introduce a factor of g, or not, depending on the particular choice of ζ.

The equation expressing conservation of an arbitrary intensive scalar, , can be written as

$$\left(\frac{\partial}{\partial t}\rho_\zeta \psi\right)_\zeta + \nabla_\zeta \cdot \left(\rho_\zeta \mathbf{V}\psi\right) + \frac{\partial}{\partial \zeta}\left(\rho_\zeta \dot{\zeta}\psi\right) = \rho_\zeta S_\psi \,. \tag{4}$$

Here

$$\dot{\zeta} \equiv \frac{D\zeta}{Dt} \tag{5}$$

is the rate of change of ζ following a particle, and S_ψ is the source or sink of ψ, per unit mass. Eq. (4) can be derived by adding up the fluxes of ψ across the boundaries of a control volume. It can also be derived by starting from the corresponding equation in a particular coordinate system, such as height, and performing a coordinate transformation. We can obtain the continuity equation in ζ-coordinates from (4), by putting $\psi \equiv 1$ and $S_\psi \equiv 0$:

$$\left(\frac{\partial \rho_\zeta}{\partial t}\right)_\zeta + \nabla_\zeta \cdot \left(\rho_\zeta \mathbf{V}\right) + \frac{\partial}{\partial \zeta}\left(\rho_\zeta \dot{\zeta}\right) = 0 \,. \tag{6}$$

By combining (4) and (6), we can obtain the advective form of the conservation equation for ψ:

$$\frac{D\psi}{Dt} = S_\psi \,, \tag{7}$$

where the Lagrangian or material time derivative is expressed by

$$\frac{D}{Dt}(\ ) = \left(\frac{\partial}{\partial t}\right)_\zeta + \mathbf{V} \cdot \nabla_\zeta + \dot{\zeta}\frac{\partial}{\partial \zeta} \,. \tag{8}$$

For example, the vertical pressure velocity,

$$\omega \equiv \frac{Dp}{Dt} , \tag{9}$$

can be expressed, using the ζ -coordinate, as

$$\begin{aligned} \omega &= \left(\frac{\partial p}{\partial t}\right)_{\zeta} + \mathbf{V} \cdot \nabla_{\zeta} p + \dot{\zeta} \frac{\partial p}{\partial \zeta} \\ &= \left(\frac{\partial p}{\partial t}\right)_{\zeta} + \mathbf{V} \cdot \nabla_{\zeta} p - \rho_{\zeta} \dot{\zeta} . \end{aligned} \tag{10}$$

The lower boundary condition, i.e., that no mass crosses the Earth's surface, is expressed by requiring that a particle that is on the Earth's surface remain there:

$$\frac{\partial \zeta_S}{\partial t} + \mathbf{V}_S \cdot \nabla \zeta_S - \dot{\zeta}_S = 0 . \tag{11}$$

In the special case in which ζ_S is independent of time and the horizontal coordinates, (11) reduces to $\dot{\zeta}_S = 0$. Eq. (11) can be derived by integration of (6) throughout the entire atmospheric column, which gives

$$\frac{\partial}{\partial t} \int_{\zeta_S}^{\zeta_T} \rho_{\zeta} \, d\zeta + \nabla \cdot \left(\int_{\zeta_S}^{\zeta_T} \rho_{\zeta} \mathbf{V} \, d\zeta \right) + \left(\rho_{\zeta} \right)_S \left(\frac{\partial \zeta_S}{\partial t} + \mathbf{V}_S \cdot \nabla \zeta_S - \dot{\zeta}_S \right) - \left(\rho_{\zeta} \right)_T \left(\frac{\partial \zeta_T}{\partial t} + \mathbf{V}_T \cdot \nabla \zeta_T - \dot{\zeta}_T \right) = 0 . \tag{12}$$

Here ζ_T is the value of ζ at the top of the model atmosphere. We allow the possibility that the top of the model is placed at a finite height and non-zero pressure. Even if the top of the model is at the "top of the atmosphere," i.e., at $p = 0$, the value of ζ_T may or may not be finite, depending on the definition of ζ . The quantity on the left-hand side of (11) is proportional to the mass flux across the Earth's surface. Similarly, $\left(\rho_{\zeta} \right)_T \left(\frac{\partial \zeta_T}{\partial t} + \mathbf{V}_T \cdot \nabla \zeta_T - \dot{\zeta}_T \right)$ represents the mass flux across the top of the atmosphere, which we assume to be zero, i.e.,

$$\frac{\partial \zeta_T}{\partial t} + \mathbf{V}_T \cdot \nabla \zeta_T - \dot{\zeta}_T = 0 . \tag{13}$$

If the top of the model is assumed to be a surface of constant ζ , which is usually the case, then (13) reduces to

$$\dot{\zeta}_T = 0 \,. \tag{14}$$

Substituting (11) and (13) into (12), we find that

$$\frac{\partial}{\partial t}\int_{\zeta_S}^{\zeta_T} \rho_\zeta \, d\zeta + \nabla \cdot \left(\int_{\zeta_T}^{\zeta_S} \rho_\zeta \mathbf{V} \, d\zeta \right) = 0 \,. \tag{15}$$

In view of (3), this is equivalent to

$$\frac{\partial p_S}{\partial t} = \frac{\partial p_T}{\partial t} - \nabla \cdot \left(\int_{\zeta_T}^{\zeta_S} \rho_\zeta \mathbf{V} \, d\zeta \right), \tag{16}$$

which is the surface pressure tendency equation. Depending on the definitions of ζ and ζ_T, it may or may not be appropriate to set $\frac{\partial p_T}{\partial t} = 0$, as an upper boundary condition. This is discussed later. Corresponding to (16), we can show that the pressure tendency on an arbitrary ζ –surface satisfies

$$\left(\frac{\partial p}{\partial t} \right)_\zeta = \frac{\partial p_T}{\partial t} - \nabla \cdot \left(\int_{\zeta_T}^{\zeta} \rho_\zeta \mathbf{V} \, d\zeta \right) + \left(\rho_\zeta \dot{\zeta} \right)_\zeta \,. \tag{17}$$

The thermodynamic equation can be written as

$$c_p \left[\left(\frac{\partial T}{\partial t} \right)_\zeta + \mathbf{V} \cdot \nabla_\zeta T + \zeta \frac{\partial T}{\partial \zeta} \right] = \omega \alpha + Q \,, \tag{18}$$

where c_p is the specific heat of air at constant pressure, α is the specific volume, and Q is the heating rate per unit mass. An alternative form of the thermodynamic equation is

$$\left(\frac{\partial \theta}{\partial t} \right)_\zeta + \mathbf{V} \cdot \nabla_\zeta \theta + \dot{\zeta} \frac{\partial \theta}{\partial \zeta} = \frac{Q}{\Pi} \,, \tag{19}$$

where

$$\Pi \equiv c_p \frac{T}{\theta} = c_p \left(\frac{p}{p_0} \right)^{\kappa} \tag{20}$$

is the Exner function. In (20), θ is the potential temperature; p_0 is a positive, constant reference pressure, usually taken to be 1000 hPa, and $\kappa \equiv \frac{R}{c_p}$, where R is the gas constant.

10.3.1 The equation of motion and the horizontal pressure-gradient force

The horizontal momentum equation can be written as

$$\left(\frac{\partial \mathbf{V}}{\partial t}\right)_\zeta + \left[f + \mathbf{k}\cdot\left(\nabla_\varsigma \times \mathbf{V}\right)\right]\mathbf{k}\times\mathbf{V} + \nabla_\varsigma K + \dot{\zeta}\frac{\partial \mathbf{V}}{\partial \zeta} = -\nabla_p \phi + \mathbf{F} . \tag{21}$$

Here $-\nabla_p \phi$ is the horizontal pressure-gradient force (hereafter abbreviated as HPGF), which is expressed as minus the gradient of the geopotential *along an isobaric surface*, and $\mathbf{F}$ is the friction vector. Also, $\mathbf{k}$ is a unit vector pointing upward, and it is important to remember that *the meaning of* $\mathbf{k}$ *is not affected by the choice of vertical coordinate system*. Similarly, $\mathbf{V}$ is the horizontal component of the velocity, and *the meaning of* $\mathbf{V}$ *is not affected by the choice of the vertical coordinate system*. Using the relation

$$\begin{aligned}\nabla_p &= \nabla_\zeta - \nabla_\zeta p \frac{\partial}{\partial p} \\ &= \nabla_\zeta + \frac{\nabla_\zeta p}{\rho_\zeta}\frac{\partial}{\partial \zeta} ,\end{aligned} \tag{22}$$

we can rewrite the HPGF as

$$-\nabla_p \phi = -\nabla_\zeta \phi - \frac{1}{\rho_\zeta}\frac{\partial \phi}{\partial \zeta}\nabla_\zeta p . \tag{23}$$

In view of (1), this can be expressed as

$$-\nabla_p \phi = -\nabla_\zeta \phi - \alpha \nabla_\zeta p . \tag{24}$$

Eq. (24) is a nice result. For $\zeta \equiv z$ it reduces to $-\nabla_p \phi = -\alpha \nabla_z p$, and for $\zeta = p$ it becomes $-\nabla_p \phi = -\nabla_p \phi$. These are both very familiar special cases.

Another useful form of the HPGF is expressed in terms of the Montgomery potential, which is defined by

$$M \equiv c_p T + \phi \,. \tag{25}$$

For the special case in which $\zeta \equiv \theta$, which will be discussed in detail later, the hydrostatic equation (1) can be written as

$$\frac{\partial M}{\partial \theta} = \Pi \,. \tag{26}$$

With the use of (25) and (26), Eq. (24) can be expressed as

$$-\nabla_p \phi = -\nabla_\zeta M + \Pi \nabla_\zeta \theta \,. \tag{27}$$

This form of the HPGF will be discussed later.

Let $q_\zeta \equiv \left(\mathbf{k} \cdot \nabla_\zeta \times \mathbf{V}\right) + f$ be the vertical component of the absolute vorticity. Note that *the meaning of* q_ζ *depends on the choice of* ζ , because the curl of the velocity is taken along a ζ -surface. Starting from the momentum equation, we can derive the vorticity equation in the form

$$\left(\frac{\partial q_\zeta}{\partial t}\right)_\zeta + \left(\mathbf{V} \cdot \nabla_\zeta\right) q_\zeta + \dot{\zeta}\frac{\partial q_\zeta}{\partial \zeta} = -q_\zeta\left(\nabla_\zeta \cdot \mathbf{V}\right) + \frac{\partial \mathbf{V}}{\partial \zeta} \times \left(\nabla_\zeta \dot{\zeta}\right) - \mathbf{k} \cdot \left[\nabla_\zeta \times \left(\nabla_p \phi\right)\right] + \mathbf{k} \cdot \left(\nabla_\zeta \times \mathbf{F}\right) . \tag{28}$$

The first term on the right-hand side of (28) represents the effects of stretching, and the second represents the effects of twisting. When the HPGF can be written as a gradient, it has no effect in the vorticity equation, because the curl of a gradient is always zero, *provided that the curl and gradient are taken along the same isosurfaces*. It is apparent from (24) and (27), however, that in general the HPGF is not simply a gradient along a ζ -surface. When the HPGF is not a gradient on a ζ -surface, it can spin up or spin down a circulation on a ζ -surface. From (24) we see that the HPGF is a pure gradient for $\zeta \equiv p$, and from (27) we see that the HPGF is a pure gradient for $\zeta \equiv \theta$. This is an advantage shared by the pressure and theta coordinates.

The *vertically integrated* HPGF has a very important property that can be used in the design of vertical differencing schemes. With the use of (1) and (3), we can rewrite (23) as follows:

$$\begin{aligned}-\rho_\zeta \mathbf{HPGF} &= -\rho_\zeta \nabla_\zeta \phi - \frac{\partial \phi}{\partial \zeta}\nabla_\zeta p \\ &= -\nabla_\zeta\left(\rho_\zeta \phi\right) + \phi \nabla_\zeta \rho_\zeta - \frac{\partial \phi}{\partial \zeta}\nabla_\zeta p \\ &= -\nabla_\zeta\left(\rho_\zeta \phi\right) - \phi \nabla_\zeta\left(\frac{\partial p}{\partial \zeta}\right) - \frac{\partial \phi}{\partial \zeta}\nabla_\zeta p \\ &= -\nabla_\zeta\left(\rho_\zeta \phi\right) - \phi \frac{\partial}{\partial \zeta}\left(\nabla_\zeta p\right) - \frac{\partial \phi}{\partial \zeta}\nabla_\zeta p \\ &= -\nabla_\zeta\left(\rho_\zeta \phi\right) - \frac{\partial}{\partial \zeta}\left(\phi \nabla_\zeta p\right).\end{aligned} \tag{29}$$

Vertically integrating with respect to mass, we find that

$$-\int_{\zeta_T}^{\zeta_S} \rho_\zeta \mathbf{HPGF}\, d\zeta = -\nabla\left(\int_{\zeta_T}^{\zeta_S} \rho_\zeta \phi\, d\zeta\right) + \left(\rho_\zeta \phi\right)_S \nabla \zeta_S - \left(\rho_\zeta \phi\right)_T \nabla \zeta_T - \phi_S\left(\nabla_\varsigma p\right)_S + \phi_T\left(\nabla_\varsigma p\right)_T . \tag{30}$$

Here we have included the $\left(\rho_\zeta \phi\right)_S \nabla \zeta_S$ and $-\left(\rho_\zeta \phi\right)_T \nabla \zeta_T$ terms to allow for the possibility that ς_T and ς_S are spatially variable. Consider a line integral of the vertically integrated HPGF, i.e., $\int_{\zeta_T}^{\zeta_S} \rho_\zeta \nabla_p \phi\, d\zeta$, along a *closed path*. Because the term $\nabla_\zeta \int_{\zeta_T}^{\zeta_S} \rho_\zeta \phi\, d\zeta$ is a gradient, its line integral must vanish. The line integral of $\phi_S \nabla p_S$ will vanish if either ϕ_S or p_S is constant along the path of integration, which is not likely with realistic geography. On the other hand, if either ϕ_T or p_T is constant along the path of integration, then the line integral of $\phi_T \nabla p_T$ will vanish, and this can easily be arranged. *We are thus motivated to choose either ϕ_T = constant or p_T = constant*. In addition, it is almost always possible (and advisable) to choose ς_T = constant.

To see how (30) plays out, let's consider two examples. For the case of pressure coordinates, with $m_p = -1$ and p_T = constant, the last two terms of (30) vanish, because they are proportional to the gradient of pressure on pressure surfaces. We get

$$-\int_{p_T}^{p_S} \mathbf{HPGF}\, dp = -\nabla\left(\int_{p_T}^{p_S} \phi\, dp\right) + \phi_S \nabla p_S$$

Note that ∇p_S is *not* the same as $\left(\nabla_p p\right)_S$ (which is equal to zero).

For the case of height coordinates, with $\rho_\zeta = \rho g$ and z_T = constant, we get

$$-\int_{z_T}^{z_S} \rho g \mathbf{HPGF}\, dz = -\nabla\left[\int_{z_T}^{z_S} \rho g \phi\, dz\right] + (\rho g \phi)_S \nabla z_S - \phi_S (\nabla_z p)_S + \phi_T (\nabla_z p)_T .$$

Swapping the limits of integration, and flipping signs to compensate, we get

$$\begin{aligned}
-\int_{z_S}^{z_T} \rho g \mathbf{HPGF}\, dz &= -\nabla\left(\int_{z_S}^{z_T} \rho g \phi\, dz\right) - (\rho g \phi)_S \nabla z_S + \phi_S (\nabla_z p)_S - \phi_T (\nabla_z p)_T \\
&= -\nabla\left(\int_{z_S}^{z_T} \rho g \phi\, dz + \phi_T p_T\right) + \phi_S\left[-(\rho g)_S \nabla z_S + (\nabla_z p)_S\right] \\
&= -\nabla\left(\int_{z_S}^{z_T} \rho g \phi\, dz - \phi_T p_T\right) + \phi_S\left[\left(\frac{\partial p}{\partial z}\right)_S \nabla z_S + (\nabla_z p)_S\right] \\
&= -\nabla\left(\int_{z_S}^{z_T} \rho g \phi\, dz - \phi_T p_T\right) + \phi_S \nabla p_S .
\end{aligned}$$

In the final line above, we have used a coordinate transformation.

We conclude that, *in the absence of topography along the path of integration, and with either* either ϕ_T = *constant or* p_T = *constant there cannot be any net spin-up or spin-down of a circulation in the region bounded by a closed path*. This conclusion is independent of the choice of vertical coordinate system. Later we will show how this important constraint can be mimicked in a vertically discrete model.

10.3.2 Vertical mass flux for a family of vertical coordinates

Konor and Arakawa (1997) derived a diagnostic equation that can be used to compute the vertical velocity, $\dot{\zeta}$, for a large family of vertical coordinates that can be expressed as functions of the potential temperature, the pressure, and the surface pressure, i.e.,

$$\zeta \equiv F(\theta, p, p_S) . \tag{31}$$

While not completely general, Eq. (31) does include a variety of interesting cases, which will be discussed below, namely:

- Pressure coordinates
- Sigma coordinates
- The hybrid sigma-pressure coordinate of Simmons and Burridge (1981)
- Theta coordinates
- The hybrid sigma-theta coordinate of Konor and Arakawa (1997).

By differentiating (31) with respect to time on a constant ζ surface, we find that

$$0=\left[\frac{\partial}{\partial t}F\left(\theta,\,p,\,p_S\right)\right]_\zeta .$$

(32)

The chain rule tells us that this is equivalent to

$$\frac{\partial F}{\partial \theta}\left(\frac{\partial \theta}{\partial t}\right)_\zeta+\frac{\partial F}{\partial p}\left(\frac{\partial p}{\partial t}\right)_\zeta+\frac{\partial F}{\partial p_S}\left(\frac{\partial p_S}{\partial t}\right)=0 .$$

(33)

Substituting from (19), (17), and (16), we obtain

$$\begin{aligned}&\frac{\partial F}{\partial \theta}\left[-\left(\mathbf{V}\cdot\nabla_\zeta\theta+\dot{\zeta}\frac{\partial \theta}{\partial \zeta}\right)+\frac{Q}{\Pi}\right]\\&+\frac{\partial F}{\partial p}\left[\frac{\partial p_T}{\partial t}-\nabla\cdot\left(\int_{\zeta_T}^{\zeta}\rho_\zeta\mathbf{V}\,d\zeta\right)+\left(\rho_\zeta\dot{\zeta}\right)_\zeta\right].\\&+\frac{\partial F}{\partial p_S}\left[\frac{\partial p_T}{\partial t}-\nabla\cdot\left(\int_{\zeta_T}^{\zeta_S}\rho_\zeta\mathbf{V}\,d\zeta\right)\right]=0 .\end{aligned}$$

(34)

This can be solved for the vertical velocity, $\dot{\zeta}$:

$$\dot{\zeta}=\frac{\frac{\partial F}{\partial \theta}\left(-\mathbf{V}\cdot\nabla_\zeta\theta+\frac{Q}{\Pi}\right)+\frac{\partial F}{\partial p}\left[\frac{\partial p_T}{\partial t}-\nabla\cdot\left(\int_{\zeta_T}^{\zeta}\rho_\zeta\mathbf{V}\,d\zeta\right)\right]+\frac{\partial F}{\partial p_S}\left[\frac{\partial p_T}{\partial t}-\nabla\cdot\left(\int_{\zeta_T}^{\zeta_S}\rho_\zeta\mathbf{V}\,dp\right)\right]}{\left\{\frac{\partial \theta}{\partial \zeta}\frac{\partial F}{\partial \theta}-\rho_\zeta\frac{\partial F}{\partial p}\right\}} .$$

(35)

Here we have ignored the possibility that the heating rate, Q , is formulated as an explicit function of $\dot{\zeta}$. In a numerical model, Q may be computed before determining the vertical velocity.

As a check, consider the special case $F\equiv p$, so that $m_\zeta=-1$, and assume that $\frac{\partial p_T}{\partial t}=0$, as would be natural for the case of pressure coordinates. Then (35) reduces to

$$\dot{p}\left(\equiv\omega\right)=-\nabla\cdot\left(\int_{p_T}^{p}\mathbf{V}\,dp\right).$$

(36)

As a second special case, suppose that $F\equiv\theta$. Then (35) becomes

$$\dot{\theta} = \frac{Q}{\Pi} . \tag{37}$$

Both of these are the expected results.

We assume that the model top is a surface of constant ζ, i.e., $\zeta_T =$ constant. Because (31) must apply at the model top, we can write

$$\left(\frac{\partial F}{\partial \theta}\right)_{\theta_T, p_T} \frac{\partial \theta_T}{\partial t} + \left(\frac{\partial F}{\partial p}\right)_{\theta_T, p_T} \frac{\partial p_T}{\partial t} + \left(\frac{\partial F}{\partial p_S}\right)_{\theta_T, p_T} \frac{\partial p_S}{\partial t} = 0 . \tag{38}$$

Suppose that $F(\theta, p, p_S)$ is chosen in such a way that $\left(\frac{\partial F}{\partial p_S}\right)_{\theta_T, p_T} = 0$. This is a natural thing to do. Then Eq. (37) simplifies to

$$\left(\frac{\partial F}{\partial \theta}\right)_{\theta_T, p_T} \frac{\partial \theta_T}{\partial t} + \left(\frac{\partial F}{\partial p}\right)_{\theta_T, p_T} \frac{\partial p_T}{\partial t} = 0 . \tag{39}$$

Consider two possibilities. If we make the top of the model an isobaric surface, so that $\frac{\partial p_T}{\partial t} = 0$, then the last term of (39) goes away, and we have the following situation: By assumption, $\left[F(\theta, p, p_S)\right]_T$ is a constant (because the top of the model is a surface of constant ζ). Also by assumption, $\left[F(\theta, p, p_S)\right]_T$ does not depend on p_S. Finally we have assumed that the top of the model is an isobaric surface. It follows that the form of $F(\theta, p, p_S)$ must be chosen so that $\left(\frac{\partial F}{\partial \theta}\right)_{\theta_T, p_T} = 0$.

As a second possibility, if we make the top of the model an isentropic surface, then $\frac{\partial \theta_T}{\partial t} = 0$, and the form of $F(\theta, p, p_S)$ must be chosen so that $\left(\frac{\partial F}{\partial p}\right)_{\theta_T, p_T} = 0$.

Further discussion is given later.

10.4 Particular vertical coordinate systems

We now discuss the following nine particular choices of vertical coordinate:

- height, z

- pressure, p
- log-pressure, z^*, which is used in many theoretical studies
- σ, defined by

$$\sigma = \frac{p - p_T}{p_S - p_T} ,$$

  which is designed to simplify the lower boundary condition
- a "hybrid," or "mix," of σ and p coordinates, used in numerous general circulation models, including the forecast model of the European Centre for Medium Range Weather Forecasts
- η, which is a modified σ coordinate, defined by

$$\eta \equiv \left(\frac{p - p_T}{p_S - p_T} \right) \eta_S ,$$

  where η_S is a time-independent function of the horizontal coordinates
- potential temperature, θ, which has many attractive properties
- entropy, $s = c_p \ln\theta$
- a hybrid sigma-theta coordinate, which behaves like σ near the Earth's surface, and like θ away from the Earth's surface.

Of these nine possibilities, all except the height coordinate and the η coordinate are members of the family of coordinates given by (31).

10.4.1 Height

In height coordinates, the hydrostatic equation is

$$\frac{\partial p}{\partial z} = -\rho g , \tag{40}$$

where $\rho \equiv 1/\alpha$ is the density. We can obtain (40) simply by flipping (2) over. For the case of the height coordinate, the pseudodensity reduces to ρg, which is proportional to the ordinary or "true" density.

The continuity equation in height coordinates is

$$\left(\frac{\partial \rho}{\partial t}\right)_z + \nabla_z \cdot (\rho \mathbf{V}) + \frac{\partial}{\partial z}(\rho w) = 0 \,. \tag{41}$$

This equation is easy to interpret, but it is mathematically complicated, in that it is nonlinear and involves the time derivative of a quantity that varies with height, namely the density.

The lower boundary condition in height coordinates is

$$\frac{\partial z_S}{\partial t} + \mathbf{V}_S \cdot \nabla z_S - w_S = 0 \,. \tag{42}$$

Normally we can assume that z_S is independent of time, but (42) can accommodate the effects of a specified time-dependent value of z_S (e.g., to represent the effects of an earthquake, or a wave on the sea surface). Because height surfaces intersect the Earth's surface, height-coordinates are relatively difficult to implement in numerical models. This complexity is mitigated somewhat by the fact that the horizontal spatial coordinates where the height surfaces meet the Earth's surface are normally independent of time.

Note that (41) and (42) are direct transcriptions of (6) and (11), respectively, with the appropriate changes in notation.

The thermodynamic energy equation is

$$c_p \rho \left(\frac{\partial T}{\partial t}\right)_z = -c_p \rho \left(\mathbf{V} \cdot \nabla_z T + w \frac{\partial T}{\partial z}\right) + \omega + \rho Q \,. \tag{43}$$

Here

$$\begin{aligned} \omega &= \left(\frac{\partial p}{\partial t}\right)_z + \mathbf{V} \cdot \nabla_z p + w \frac{\partial p}{\partial z} \\ &= \left(\frac{\partial p}{\partial t}\right)_z + \mathbf{V} \cdot \nabla_z p - \rho g w \,. \end{aligned} \tag{44}$$

By using (44) in (43), we find that

$$c_p \rho \left(\frac{\partial T}{\partial t}\right)_z = -c_p \rho \mathbf{V} \cdot \nabla_z T - \rho w c_p \left(\Gamma_d - \Gamma\right) + \left[\left(\frac{\partial p}{\partial t}\right)_z + \mathbf{V} \cdot \nabla_z p\right] + \rho Q \,, \tag{45}$$

where the actual lapse rate and the dry-adiabatic lapse rate are given by

$$\Gamma \equiv -\frac{\partial T}{\partial z}, \tag{46}$$

and

$$\Gamma_d \equiv -\frac{g}{c_p}. \tag{47}$$

respectively. Eq. (45) is awkward because it involves the time derivatives of both T and p. The time derivative of the pressure can be eliminated by using the height-coordinate version of (17), which is

$$\left(\frac{\partial p}{\partial t}\right)_z = -g\nabla_z \cdot \int_z^\infty (\rho\mathbf{V})\,dz + g\rho(z)w(z) + \frac{\partial p_T}{\partial t} \tag{48}$$

Substitution into (45) gives

$$\begin{aligned} c_p\rho\left(\frac{\partial T}{\partial t}\right)_z &= -c_p\rho\mathbf{V}\cdot\nabla_z T - \rho w c_p\left(\Gamma_d - \Gamma\right) \\ &\quad + \left[-g\nabla_z \cdot \int_z^\infty (\rho\mathbf{V})\,dz + g\rho(z)w(z) + \frac{\partial p_T}{\partial t}\right] + \mathbf{V}\cdot\nabla_z p + \rho Q\,. \end{aligned} \tag{49}$$

According to (49), the time rate of change of the temperature at a given height is influenced by the motion field through a deep layer. An alternative, considerably simpler form of the thermodynamic energy equation in height coordinates is

$$\left(\frac{\partial \theta}{\partial t}\right)_z = -\left(\mathbf{V}\cdot\nabla_z\theta + w\frac{\partial\theta}{\partial z}\right) + \frac{Q}{\Pi}. \tag{50}$$

In quasi-static models based on height coordinates, the equation of vertical motion is replaced by the hydrostatic equation, in which w does not even appear. How then can we compute w? The height coordinate is not a member of the family of schemes defined by (31), and so (35), the formula for the vertical mass flux derived from (31), does not apply. Instead, w is computed using "Richardson's equation," which is an expression of the physical fact that hydrostatic balance applies not just at a particular instant, but continuously through time. Richardson's equation is actually closely analogous to (35), but somewhat more complicated. The derivation of Richardson's equation is also more complicated than the derivation of (35). Here it comes:

The equation of state is

$$p = \rho R T \, . \tag{51}$$

Logarithmic differentiation of (51) with respect to time gives

$$\frac{1}{p}\left(\frac{\partial p}{\partial t}\right)_z = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial t}\right)_z + \frac{1}{T}\left(\frac{\partial T}{\partial t}\right)_z \, . \tag{52}$$

The time derivatives can be eliminated by using (41), (45) and (48). After some manipulation, we find that

$$c_p T \frac{\partial}{\partial z}(\rho w) + \rho w\left[g\frac{c_v}{R} + c_p\left(\Gamma_d - \Gamma\right)\right] = \left(-c_p \rho \mathbf{V}\cdot\nabla_z T + \mathbf{V}\cdot\nabla_z p\right) - c_p T \nabla_z \cdot (\rho \mathbf{V}) + g\frac{c_v}{R}\nabla_z \cdot \int_z^\infty (\rho \mathbf{V})\, dz + \rho Q \, , \tag{53}$$

where

$$c_v \equiv c_p - R \tag{54}$$

is the specific heat of air at constant volume.

Eq. (53) can be simplified considerably as follows. Expand the vertical derivative term using the product rule:

$$c_p T \frac{\partial(\rho w)}{\partial z} = \rho c_p T \frac{\partial w}{\partial z} + w c_p T \frac{\partial \rho}{\partial z} \, , \tag{55}$$

Logarithmic differentiation of (51) with respect to height gives

$$\frac{1}{p}\frac{\partial p}{\partial z} = \frac{1}{\rho}\frac{\partial \rho}{\partial z} + \frac{1}{T}\frac{\partial T}{\partial z} \, , \tag{56}$$

which is equivalent to

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{\rho g}{p} + \frac{\Gamma}{T}$$
$$= \frac{1}{T}\left(-\frac{g}{R} + \Gamma\right). \tag{57}$$

Substitute (57) into (55) to obtain

$$c_p T \frac{\partial(\rho w)}{\partial z} = \rho c_p T \frac{\partial w}{\partial z} + \rho w c_p \left(-\frac{g}{R} + \Gamma\right). \tag{58}$$

Finally, substitute (58) into (53), and combine terms, to obtain

$$\boxed{\rho c_p T \frac{\partial w}{\partial z} = \left(-c_p \rho \mathbf{V} \cdot \nabla_z T + \mathbf{V} \cdot \nabla_z p\right) - c_p T \nabla_z \cdot (\rho \mathbf{V}) + g \frac{c_v}{R} \nabla_z \cdot \int_z^\infty (\rho \mathbf{V})\, dz + \rho Q}\,. \tag{59}$$

This beast is Richardson's equation. It can be solved as a linear first-order ordinary differential equation for $w(z)$, given a lower boundary condition and the information needed to compute the various terms on the right-hand side, which involve both the mean horizontal motion and the heating rate, as well as various horizontal derivatives. A physical interpretation of (59) is that the vertical motion is whatever it takes to maintain hydrostatic balance through time despite the fact that the various processes represented on the right-hand side of (59) may tend to upset that balance.

As a very simple illustration of the use of (59), suppose that we have horizontally uniform heating but no horizontal motion. Then (59) drastically simplifies to

$$c_p T \frac{\partial w}{\partial z} = Q\ . \tag{60}$$

If the lower boundary is flat, so that

$$w = 0 \ \text{ at } \ z = 0\,, \tag{61}$$

we obtain

$$w(z) = \int_0^{\tilde{z}} \frac{Q}{c_p T}\, dz\,, \tag{62}$$

i.e., heating (cooling) below a given level induces rising (sinking) motion at that level. The rising motion induced by heating below a given level can be interpreted as a manifestation of the upward movement of air particles as the air expands above the rigid lower boundary.

The complexity of Richardson's equation has discouraged the use of height coordinates in quasi-static models; one of the very few exceptions was the early NCAR GCM (Kasahara and Washington, 1967). We are now entering an era of non-hydrostatic global models, in which use of the height coordinate may become more common.

10.4.2 Pressure

The hydrostatic equation in pressure coordinates has already been stated; it is (2). The pseudo-density is simply unity, since (3) reduces to

$$\rho_p = 1 . \tag{63}$$

Here we drop the minus sign that was used in (3). The continuity equation in pressure coordinates is relatively simple; it is linear and does not involve a time derivative. Eq. (6) reduces to

$$\nabla_p \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0 . \tag{64}$$

On the other hand, the lower boundary condition is complicated in pressure coordinates:

$$\frac{\partial p_S}{\partial t} + \mathbf{V}_S \cdot \nabla p_S - \omega_S = 0 . \tag{65}$$

Recall that p_S can be predicted using the surface pressure-tendency equation, (16). Substitution from (16) into (65) gives

$$\omega_S = \frac{\partial p_T}{\partial t} - \nabla \cdot \left(\int_{p_T}^{p_S} \mathbf{V}\, dp \right) + \mathbf{V}_S \cdot \nabla p_S , \tag{66}$$

which can be used to diagnose ω_S . The fact that pressure surfaces intersect the ground at locations that change with time (unlike height surfaces), means that models that use pressure coordinates are complicated. Largely for this reason, pressure coordinates are hardly ever used in numerical models. One of the few exceptions is the early and short-lived general circulation model developed by Leith at the Lawrence National Laboratory (now the Lawrence Livermore National Laboratory).

With the pressure coordinate, we can write

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right)\right]_p = -\frac{R}{p}\left(\frac{\partial T}{\partial t}\right)_p . \tag{67}$$

This allows us to eliminate the temperature in favor of the geopotential, which is often done in theoretical studies.

10.4.3 Log-pressure

Obviously a surface of constant p is also a surface of constant $\ln p$. Nevertheless, the equations take different forms in the p and $\ln p$ coordinate systems.

Let T_0 be a constant reference temperature, and $H \equiv \frac{RT_0}{g}$ the corresponding scale height. *Define* the "log-pressure coordinate" z^* by the differential relationship

$$dz^* = -Hd(\ln p) = -H\frac{dp}{p} . \tag{68}$$

Note that z^* has the units of length (i.e., height), and that

$$dz^* = dz \text{ when } T(p) = T_0 . \tag{69}$$

Although generally $z \neq z^*$, we can force $z(p = p_S) = z^*(p = p_S)$. From (68), we see that

$$\frac{\partial \phi^*}{\partial p} = -\frac{RT_0}{p} , \tag{70}$$

where

$$\phi^* \equiv gz^* . \tag{71}$$

We also have the hydrostatic equation in the form

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p} . \tag{72}$$

Subtracting (70) from (72), we obtain a useful form of the hydrostatic equation:

$$\frac{\partial\left(\phi-\phi^*\right)}{\partial p}=-\frac{R\left(T-T_0\right)}{p}. \tag{73}$$

Since ϕ^* and T_0 are independent of time, we see that

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right)_{z^*}=-\frac{R}{p}\left(\frac{\partial T}{\partial t}\right)_{z^*}. \tag{74}$$

10.4.4 The σ –coordinate

The σ -coordinate of Phillips (1957) is defined by

$$\sigma \equiv \frac{p-p_T}{\pi}, \tag{75}$$

where

$$\pi \equiv p_S - p_T, \tag{76}$$

which is independent of height. Obviously,

$$\sigma_S = 1 \text{ and } \sigma_T = 0. \tag{77}$$

Notice that the top of the model is an isobaric surface, assuming that p_T = constant. Phillips (1957) took $p_T = 0$.

Inverting (75), we can write

$$p = p_T + \sigma\pi.$$

For a fixed value of σ,

$$dp = \sigma d\pi, \tag{78}$$

where the differential can represent a fluctuation in either time the horizontal space, *with a fixed value of* σ. Also,

$$\frac{\partial}{\partial p}(\ )=\frac{1}{\pi}\frac{\partial}{\partial \sigma}(\ ). \tag{79}$$

Here the differentials are evaluated at fixed horizontal position and time.

The pseudodensity in σ-coordinates is

$$\rho_\sigma=\pi , \tag{80}$$

which is independent of height. Here we choose not to use the minus sign in (3). The continuity equation in σ-coordinates can therefore be written as

$$\frac{\partial \pi}{\partial t}+\nabla_\sigma\cdot(\pi\mathbf{V})+\frac{\partial(\pi\dot{\sigma})}{\partial \sigma}=0 . \tag{81}$$

Although this equation does contain a time derivative, the differentiated quantity, π , is independent of height, which makes (81) considerably simpler than (6) or (41).

The lower boundary condition in σ-coordinates is very simple:

$$\dot{\sigma}=0 \text{ at } \sigma=1 . \tag{82}$$

This simplicity was in fact Phillips' motivation for the invention of σ-coordinates. The upper boundary condition is similar:

$$\dot{\sigma}=0 \text{ at } \sigma=0 . \tag{83}$$

The continuity equation in σ-coordinates plays a dual role. First, it is used to predict π. This is done by integrating (81) through the depth of the vertical column and using the boundary conditions (82) and (83), to obtain the surface pressure-tendency equation in the form

$$\frac{\partial \pi}{\partial t}=-\nabla\cdot\left(\int_0^1 \pi\mathbf{V}d\sigma\right). \tag{84}$$

The continuity equation is also used to determine $\pi\dot{\sigma}$. Once $\frac{\partial \pi}{\partial t}$ has been evaluated using (84), which does not involve $\pi\dot{\sigma}$, we can substitute back into (81) to obtain

$$\frac{\partial}{\partial\sigma}(\pi\dot{\sigma}) = \nabla\cdot\left(\int_0^1 \pi\mathbf{V}d\sigma\right) - \nabla_\sigma\cdot(\pi\mathbf{V}).$$

(85)

This can be integrated vertically to obtain $\pi\dot{\sigma}$ as a function of σ, starting from either the Earth's surface or the top of the atmosphere, and using the appropriate boundary condition at the top or bottom. The same result is obtained regardless of the direction of integration. This method gives a result that is consistent with Eq. (35).

The hydrostatic equation in -coordinates is simply

$$\frac{1}{\pi}\frac{\partial\phi}{\partial\sigma} = -\alpha\,,$$

(86)

which is closely related to (2). Finally, the horizontal pressure-gradient force takes a relatively complicated form:

$$\mathbf{HPGF} = -\sigma\alpha\nabla\pi - \nabla_\sigma\phi\,,$$

(87)

which can easily be obtained from (24). Using the hydrostatic equation, (86), we can rewrite this as

$$\mathbf{HPGF} = \sigma\left(\frac{1}{\pi}\frac{\partial\phi}{\partial\sigma}\right)\nabla\pi - \nabla_\sigma\phi\,.$$

(88)

Rearranging, we find that

$$\begin{aligned}\pi(\mathbf{HPGF}) &= \sigma\frac{\partial\phi}{\partial\sigma}\nabla\pi - \pi\nabla_\sigma\phi \\ &= \left[\frac{\partial(\sigma\phi)}{\partial\sigma} - \phi\right]\nabla\pi - \pi\nabla_\sigma\phi \\ &= \frac{\partial(\sigma\phi)}{\partial\sigma}\nabla\pi - \nabla_\sigma(\pi\phi)\,.\end{aligned}$$

(89)

This is a special case of (29).

Consider the two contributions to the HPGF when evaluated near a mountain, as

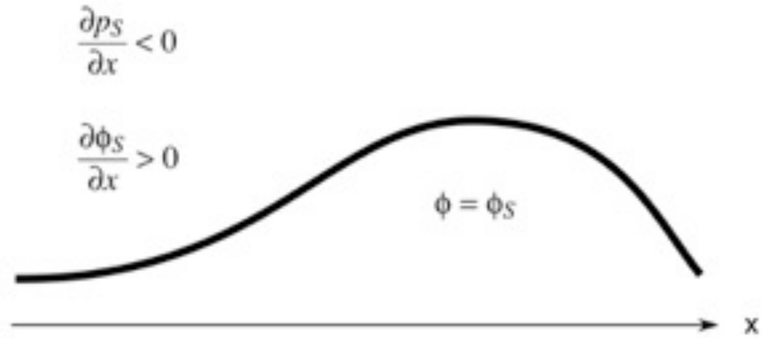

Fig. 10.1: Sketch illustrating the opposing terms of the horizontal pressure gradient force as measured in σ-coordinates.

illustrated in Fig. 10.1. Near steep topography, the spatial variations of p_S and the near-surface value of ϕ along a σ-surface are strong and of opposite sign. For example, moving uphill p_S decreases while ϕ_S increases. As a result, the two terms on the right-hand side of (86) are individually large and opposing, and the HPGF is the relatively small difference between them- a dangerous situation. In numerical models based on the σ-coordinate, near steep mountains the relatively small discretization errors in the individual terms of the right-hand side of (86) can be as large as the HPGF.

This may appear to be an issue mainly with horizontal differencing, because the HPGF involves horizontal derivatives, but vertical differencing also comes in. To see how, consider Fig. 10.2. At the point O, the $\sigma = \sigma *$ and $p = p *$ surfaces intersect. As we move away from point O, the two surfaces separate. By a coordinate transformation, we can write

$$\begin{aligned}\mathbf{HPGF} &= -\nabla_p \phi \\ &= -\nabla_\sigma \phi + \frac{\partial \phi}{\partial p} \nabla_\sigma p \ .\end{aligned} \tag{90}$$

This second line of (90) expresses the $HPGF$ in terms of both the horizontal change in ϕ along a σ-surface, say between two neighboring horizontal grid points (mass points), and the vertical change in ϕ between neighboring model layers. The latter depends, hydrostatically, on the temperature. Using hydrostatics, the ideal gas law, and the definition of σ, we can rewrite (90) as

$$\mathbf{HPGF} = -\nabla_\sigma \phi - \left(\frac{RT}{p} \right) \sigma \nabla \pi \ . \tag{91}$$

Compare with (87).

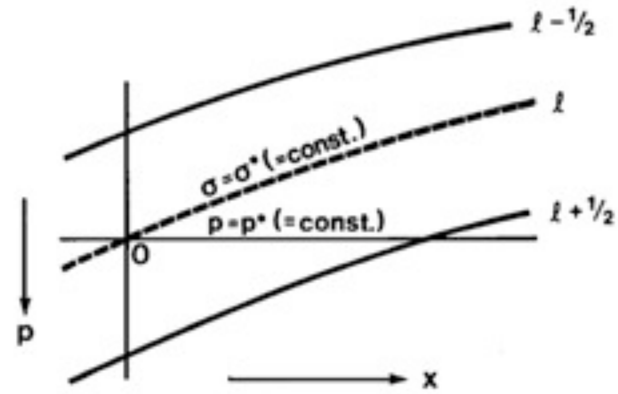

Fig. 10.2: Sketch illustrating the pressure-gradient force as seen in σ -coordinates and pressure coordinates.

If the σ -surfaces are very steeply tilted relative to constant height surfaces, which can happen especially near steep mountains, the temperature needed on the right-hand side of (90) will be representative of two or more σ -layers, rather than a single layer. If the temperature is changing rapidly with height, this can lead to large errors. It can be shown that the problem is minimized if the model has sufficiently high horizontal resolution relative to its vertical resolution (Janjic, 1977; Mesinger, 1982; Mellor et al., 1994), i.e.,

$$\frac{\delta\sigma}{\delta x} \geq \frac{\left| \left(\frac{\delta\phi}{\delta x} \right)_\sigma \right|}{\left| \left(\frac{\delta\phi}{\delta\sigma} \right)_x \right|} \ . \tag{92}$$

The numerator of the right-hand side of (92) increases when the terrain is steep. The denominator increases when T is warm, i.e., near the surface. The inequality (92) means that $\delta\sigma$ must be coarse enough for a given δx . Increasing the vertical resolution without a corresponding increase in the horizontal resolution can cause problems.

The Lagrangian time derivative of pressure can be expressed in σ -coordinates as

$$\omega \equiv \frac{Dp}{Dt} = \left(\frac{\partial p}{\partial t} \right)_\sigma + \mathbf{V} \cdot \nabla_\sigma p + \dot{\sigma} \frac{\partial p}{\partial \sigma}$$
$$= \sigma \left(\frac{\partial \pi}{\partial t} + \mathbf{V} \cdot \nabla \pi \right) + \pi \dot{\sigma} \; .$$

10.4.5 Hybrid sigma-pressure coordinates

The advantage of the sigma coordinate is realized in the lower boundary condition. The disadvantage, in terms of the complicated and poorly behaved pressure-gradient force, is realized at all levels. This has motivated the use of hybrid coordinates that reduce to sigma at the lower boundary, and become pure pressure-coordinates at higher levels. In principle there are many ways of doing this. The most widely cited reference on this topic is the paper of Simmons and Burridge (1981). They recommended the coordinate

$$\xi(p, p_S) = \frac{p}{p_0}\left(1 - \frac{p}{p_S}\right) + \left(\frac{p}{p_S}\right)^2 , \tag{93}$$

where p_0 is a positive constant. It can be demonstrated that ξ is monotonic with pressure, provided that $p_0 > p_S / 2$. Inspection of (93) shows that

$$\xi = 0 \text{ for } p = 0 \text{, and } \xi = 1 \text{ for } p = p_S \; . \tag{94}$$

It can be shown that ξ–surfaces are nearly parallel to isobaric surfaces in the upper troposphere and stratosphere, despite possible variations of the surface pressure in the range ~1000 mb to ~500 mb. When we evaluate the HPGF with the ξ-coordinate, there are still two terms, as with the σ-coordinate, but above the lower troposphere one of the terms is strongly dominant.

10.5 The η-coordinate

As a solution to the problem with the HPGF in σ-coordinates, Mesinger and Janjic (1985) proposed the η-coordinate, which has been used operationally at NCEP (the National Centers for Environmental Prediction):

$$\eta \equiv \sigma \eta_S , \tag{95}$$

where

$$\eta_S = \frac{p_{rf}(z_S) - p_T}{p_{rf}(0) - p_T} \, .$$

(96)

Whereas $\sigma = 1$ at the Earth's surface, Eq. (95) shows that $\eta = \eta_S$ at the Earth's surface. According to (96), $\eta_S = 1$ (just as $\sigma = 1$) if $z_S = 0$. Here $z_S = 0$ is chosen to be at or near "sea level." The function $p_{rf}(z_S)$ is pre-specified as a typical surface pressure for $z = z_S$. Because z_S depends on the horizontal coordinates, $p_{rf}(z_S)$ does too, and so, therefore, does η_S. In fact, after choosing $p_{rf}(z_S)$ and $z_S(x,y)$, one can make a map of $\eta_S(x,y)$, and of course *this map is independent of time*.

When we build a σ -coordinate model, we must specify (i.e., choose) values of σ to serve as layer-edges and/or layer centers. These values are constant in the horizontal and time. Similarly, when we build an η-coordinate model, we must specify fixed values of η to serve as layer edges and/or layer centers. The values of η to be chosen include the possible values of η_S. This means that *only a finite number of discrete (and constant) values of η_S are permitted*; the number increases as the vertical resolution of the model increases. Mountains must come in a few discrete sizes, like off-the-rack clothing! This is sometimes called the "step-mountain"

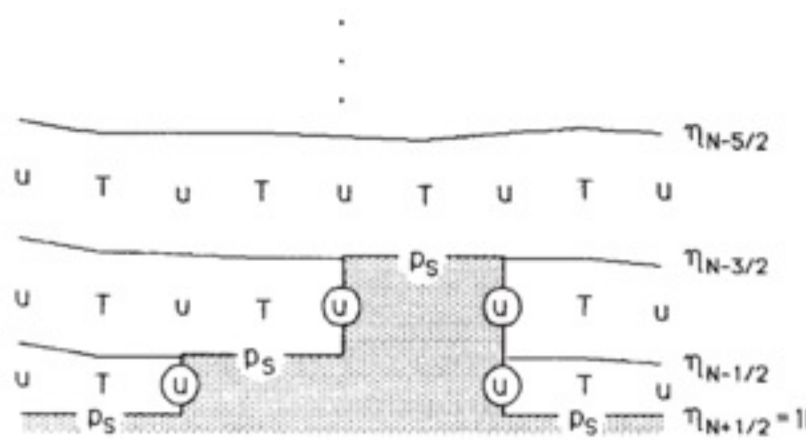

Fig. 10.3: Sketch illustrating the η-coordinate.

approach. Fig. 10.3 shows how the η -coordinate works near mountains. Note that, unlike σ -surfaces, η -surfaces are nearly flat, in the sense that they are close to being isobaric surfaces. The circled u-points have $u = 0$, as a boundary condition on the sides of the mountains.

In -coordinates, the HPGF still consists of two terms:

$$-\nabla_p \phi = -\nabla_\eta \phi - \alpha \nabla_\eta p \,. \tag{97}$$

Because the η -surfaces are nearly flat, however, these two terms are each comparable in magnitude to the HPGF itself, even near mountains, so the problem of near-cancellation does not occur.

10.5.1 Potential temperature

The potential temperature is defined by

$$\theta \equiv T\left(\frac{p_0}{p}\right)^{\kappa} \,. \tag{98}$$

The potential temperature increases upwards in a statically stable atmosphere, so that there is a monotonic relationship between θ and z. Note, however, that potential temperature cannot be used as a vertical coordinate when static instability occurs, and that the vertical resolution of a θ -coordinate model becomes very poor when the atmosphere is close to neutrally stable.

Potential temperature coordinates have highly useful properties that have been recognized for many years, and have become more widely appreciated during the past decade or so. In the absence of heating, potential temperature is conserved following a particle. This means that the vertical motion in θ -coordinates is proportional to the heating rate:

$$\dot{\theta} = \frac{\theta}{c_p T} Q \,; \tag{99}$$

in the absence of heating, there is "no vertical motion," from the point of view of θ -coordinates; we can also say that, in the absence of heating, a particle that is on a given θ -surface remains on that surface. Eq. (99) is an expression of the thermodynamic energy equation in θ -coordinates. In fact, θ -coordinates provide an especially simple pathway for the derivation of many important results, including the conservation equation for the Ertel potential vorticity. In addition, θ -coordinates prove to have some important advantages for the design of numerical models (e.g., Eliassen and Raustein, 1968; Bleck, 1973; Johnson and Uccellini, 1983; Hoskins et al. 1985; Hsu and Arakawa, 1990).

The continuity equation in θ -coordinates is given by

$$\left(\frac{\partial \rho_\theta}{\partial t}\right)_\theta + \nabla_\theta \cdot \left(\rho_\theta \mathbf{V}\right) + \frac{\partial}{\partial \theta}\left(\rho_\theta \dot{\theta}\right) = 0 \,, \tag{100}$$

which is a direct transcription of (6). Note, however, that $\dot{\theta}=0$ in the absence of heating; in such case, (100) reduces to

$$\left(\frac{\partial \rho_\theta}{\partial t}\right)_\theta + \nabla_\theta \cdot \left(\rho_\theta \mathbf{V}\right) = 0\,, \tag{101}$$

which is closely analogous to the continuity equation of a shallow-water model.

The lower boundary condition in θ-coordinates is

$$\frac{\partial \theta_S}{\partial t} + \mathbf{V}\cdot\nabla\theta_S - \dot{\theta}_S = 0\,. \tag{102}$$

This equation can be used to predict θ_S. The complexity of the lower boundary condition in θ-coordinates is one of its chief drawbacks. This will be discussed further below.

For the case of θ-coordinates, the hydrostatic equation, (1), reduces to

$$\frac{\partial \phi}{\partial \theta} = \alpha \frac{\partial p}{\partial \theta}\,. \tag{103}$$

"Logarithmic differentiation" of (97) gives

$$\frac{d\theta}{\theta} = \frac{dT}{T} - \kappa \frac{dp}{p}\,. \tag{104}$$

It follows that

$$\alpha \frac{\partial p}{\partial \theta} = c_p \frac{\partial T}{\partial \theta} - c_p \frac{T}{\theta}\,. \tag{105}$$

Substitution of (105) into (103) gives

$$\frac{\partial M}{\partial \theta} = \Pi\,, \tag{106}$$

where M was defined in (25).

The HPGF in θ-coordinates is

$$\mathbf{HPGF} = -\alpha\nabla_\theta p - \nabla_\theta \phi .$$

(107)

From (104) it follows that

$$\nabla_\theta p = c_p\left(\frac{p}{RT}\right)\nabla_\theta T .$$

(108)

Substitution of (107) into (106) gives

$$\mathbf{HGPF} = -\nabla_\theta M .$$

(109)

Of course, θ -surfaces can intersect the lower boundary, but we can consider these to

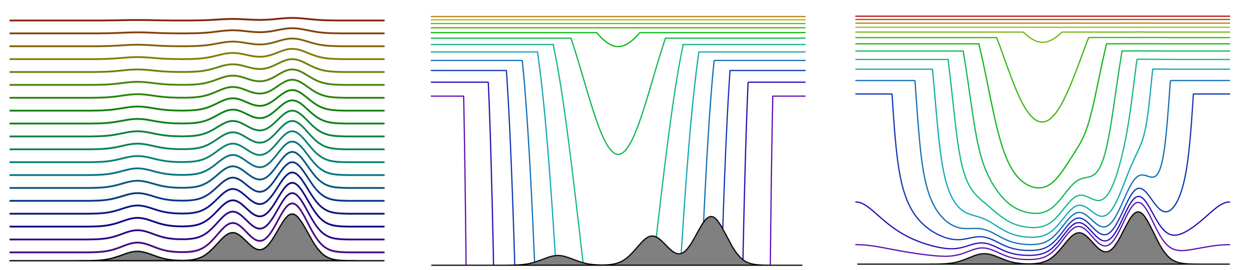

Fig. 10.4: Coordinate surfaces with topography: Left, the σ -coordinate. Center, the θ -coordinate. Right, a hybrid σ-θ coordinate.

follow the boundary, by defining imaginary "massless layers," as shown in Fig. 10.4. Since no mass resides between the θ-surfaces in the portion of the domain where they "touch the Earth's surface," no harm is done by this fantasy.

Obviously, a model that follows the massless-layer approach has to be able to deal with avoid producing negative mass, e.g., through the use of flux-corrected transport. This practical difficulty has led most modelers to avoid θ-coordinates up to this time.

The massless layer approach leads us to use values of θ that are colder than any actually present in an atmospheric column, particularly in the tropics of a global model. The coldest possible value of θ is zero Kelvin. Consider the lower boundary condition on the hydrostatic equation, (106). We can write

$$M(\theta) - M(0) = \int_0^{\theta} \Pi(\theta')\, d\theta' ,$$

where θ' is a dummy variable of integration. From the definition of M, we have $M(0) = \phi_S$. For "massless" portion of the integral, the integrand, $\Pi(\theta')$, is just a constant, namely Π_S, i.e., the surface value of Π. We can therefore write

$$\begin{aligned} M(\theta) - \phi_S &= \int_0^{\theta_S} \Pi(\theta')\, d\theta' + \int_{\theta_S}^{\theta} \Pi(\theta')\, d\theta' \\ &= \Pi_S \theta_S + \int_{\theta_S}^{\theta} \Pi(\theta')\, d\theta' \\ &= c_p T_S + \int_{\theta_S}^{\theta} \Pi(\theta')\, d\theta' . \end{aligned}$$

It follows that

$$M(\theta) = c_p T + \phi_S + \int_{\theta_S}^{\theta} \Pi(\theta')\, d\theta' ,$$

as expected.

The dynamically important isentropic potential vorticity, q, is easily constructed in θ-coordinates, since it involves the curl of $\mathbf{V}$ on a θ-surface:

$$q \equiv \left(\mathbf{k} \cdot \nabla_\theta \times \mathbf{V} + f\right) \frac{\partial \theta}{\partial p} . \tag{110}$$

The available potential energy is also easily obtained, since it involves the distribution of pressure on θ-surfaces.

10.5.2 Entropy

The entropy coordinate is very similar to the θ-coordinate. We define the entropy by

$$s = c_p \ln \theta , \tag{111}$$

so that

$$ds = c_p \frac{d\theta}{\theta} . \tag{112}$$

The hydrostatic equation can then be written as

$$\frac{\partial M}{\partial s} = T \, . \tag{113}$$

This is a particularly attractive form because the "thickness" is simply given by the temperature.

10.5.3 Hybrid $\sigma - \theta$ coordinates

Konor and Arakawa (1997) discuss a hybrid vertical coordinate, ζ, that reduces to θ away from the surface, and to σ near the surface. This hybrid coordinate is a member of the family of schemes given by (31). It is designed to combine the strengths of θ and σ coordinates, while avoiding their weaknesses. Hybrid coordinates have also been considered by other authors, e.g., Johnson and Uccellini (1983) and Zhu et al. (1992).

To specify the scheme, we must choose the function $F(\theta, p, p_S)$ that appears in (31). Following Konor and Arakawa (1997), define

$$\zeta = F(\theta, p, p_S) \equiv f(\sigma) + g(\sigma)\theta \, , \tag{114}$$

where $\sigma \equiv \sigma(p, p_S)$ is a modified sigma coordinate, defined so that it is (as usual) a constant at the Earth's surface, and (not as usual) increases upwards, e.g.,

$$\sigma \equiv \frac{p_S - p}{p_S} \, . \tag{115}$$

If we specify $f(\sigma)$ and $g(\sigma)$, then the hybrid coordinate is fully determined.

We require, of course, that ζ itself increases upwards, so that

$$\frac{\partial \zeta}{\partial \sigma} > 0 \, . \tag{116}$$

We also require that

$$\zeta = \text{constant for } \sigma = \sigma_S \, , \tag{117}$$

which means that ζ is σ-like at the Earth's surface, and that

$$\zeta = \theta \text{ for } \sigma = \sigma_T , \tag{118}$$

which means that ζ becomes θ at the model top (or lower). These conditions imply, from (114), that

$$g(\sigma) \to 0 \text{ as } \sigma \to \sigma_S , \tag{119}$$

$$f(\sigma) \to 0 \text{ and } g(\sigma) \to 1 \text{ as } \sigma \to \sigma_T . \tag{120}$$

Now substitute (114) into (116), to obtain

$$\frac{df}{d\sigma} + \theta \frac{dg}{d\sigma} + g \frac{\partial \theta}{\partial \sigma} > 0 . \tag{121}$$

This is the requirement that ζ increases monotonically upward. Any choices for *f* and *g* that satisfy (119) - (121) can be used to define the hybrid coordinate.

Here is a way to do that: First, choose $g(\sigma)$ so that it is a monotonically increasing function of height, i.e.,

$$\frac{dg}{d\sigma} > 0 \text{ for all } \sigma . \tag{122}$$

We also choose $g(\sigma)$ so that the conditions (119) - (120) are satisfied. Obviously there are many possible choices for $g(\sigma)$ that will meet these requirements.

Next, define $\theta_{\min}$ and $\left(\frac{\partial \theta}{\partial \sigma} \right)_{\min}$ as lower bounds on θ and $\frac{\partial \theta}{\partial \sigma}$, respectively, i.e.,

$$\theta > \theta_{\min} \text{ and } \frac{\partial \theta}{\partial \sigma} > \left(\frac{\partial \theta}{\partial \sigma} \right)_{\min} . \tag{123}$$

When we choose $\theta_{\min}$ *the value of, we are saying that we have no interest in simulating situations in which* θ *is actually colder than* $\theta_{\min}$. For example, we could choose $\theta_{\min}$ =10K. This is not necessarily an ideal choice, for reasons to be discussed below, but we can be sure that in our simulations will exceed 10 K everywhere at all times, unless the model is in the final throes of blowing up. *Similarly, when we choose the value of* $\left(\frac{\partial\theta}{\partial\sigma}\right)_{\min}$ *, we are saying that we have no interest in simulating situations in which* $\frac{\partial\theta}{\partial\sigma}$ *is actually less stable (or more unstable) than* $\left(\frac{\partial\theta}{\partial\sigma}\right)_{\min}$. We can choose $\left(\frac{\partial\theta}{\partial\sigma}\right)_{\min} < 0$, i.e., a value of $\left(\frac{\partial\theta}{\partial\sigma}\right)_{\min}$ that corresponds to a statically unstable sounding. Further discussion is given below.

Now, with reference to the *inequality* (121), we write the following *equation*:

$$\frac{df}{d\sigma} + \frac{dg}{d\sigma}\theta_{\min} + g\left(\frac{\partial\theta}{\partial\sigma}\right)_{\min} = 0 . \tag{124}$$

Recall that $g(\sigma)$ will be specified in such a way that (122) is satisfied. You should be able to see that if the *equality* (123) is satisfied, then the *inequality* (120) will also be satisfied, i.e. ζ , will increase monotonically upward. *This will be true even if the sounding is statically unstable in some regions, provided that* (123) *is satisfied*.

Eq. (124) is a first-order ordinary differential equation for $f(\sigma)$, which must be solved subject to the boundary condition (120).

That's all there is to it. Amazingly, the scheme does not involve any "if-tests." It is simple and fairly flexible.

The vertical velocity is obtained using (35).

10.5.4 Summary of vertical coordinate systems

Table 10.1 summarizes key properties of some important vertical coordinate systems. All of the systems discussed here (with the exception of the entropy coordinate) have been used in

Coordinate	Hydrostatics	HPGF	Vertical velocity	Continuity	LBC
z	$\frac{\partial p}{\partial z}=-\rho g$	$-\alpha\nabla_z p$	$w\equiv\frac{Dz}{Dt}$	$\frac{\partial\rho}{\partial t}+\nabla_z\cdot(\rho\mathbf{V})+\frac{\partial(\rho w)}{\partial z}=0$	$\mathbf{V}_S\cdot\nabla z_S-w_S=0$
p	$\frac{\partial\phi}{\partial p}=-\alpha$	$-\nabla_p\phi$	$\omega\equiv\frac{Dp}{Dt}$	$\nabla_p\cdot(\mathbf{V})+\frac{\partial\omega}{\partial p}=0$	$\frac{\partial p_S}{\partial t}+\mathbf{V}_S\cdot\nabla p_S-\omega_S=0$
$z^*\equiv-H\ln\left(\frac{p}{p_0}\right)$	$\frac{\partial z}{\partial z^*}=-\frac{T}{T_0}$	$-\nabla_{z^*}\phi$	$w\equiv\frac{Dz^*}{Dt}=-\frac{H\omega}{p}$	$\nabla_{z^*}\cdot\mathbf{V}+\frac{\partial w^*}{\partial z^*}-\frac{w^*}{H}=0$	$\frac{\partial z_S^*}{\partial t}+\mathbf{V}_S\cdot\nabla z_S^*-w_S^*=0$
$\sigma\equiv\frac{p-p_T}{\pi}$	$\frac{1}{\pi}\frac{\partial\phi}{\partial\sigma}=-\alpha$	$-\nabla_\sigma\phi-\sigma\alpha\pi$	$\dot\sigma\equiv\frac{D\sigma}{Dt}$	$\frac{\partial\pi}{\partial t}+\nabla_\sigma\cdot(\pi\mathbf{V})+\frac{\partial(\pi\dot\sigma)}{\partial z}=0$	$-\dot\sigma_S=0$
θ	$\frac{\partial M}{\partial\theta}=\pi$	$-\nabla_\theta M$	$\dot\theta\equiv\frac{D\theta}{Dt}$	$\frac{\partial m}{\partial t}+\nabla_\theta\cdot(m\mathbf{V})+\frac{\partial(m\dot\theta)}{\partial\theta}=0$	$\frac{\partial\theta_S}{\partial t}+\mathbf{V}_S\cdot\nabla\theta_S-\dot\theta_S=0$
s	$\frac{\partial\psi}{\partial s}=T$	$-\nabla_s M$	$\dot s\equiv\frac{Ds}{Dt}$	$\frac{\partial\mu}{\partial t}+\nabla_s\cdot(\mu\mathbf{V})+\frac{\partial(\mu\dot s)}{\partial s}=0$	$\frac{\partial s_S}{\partial t}+\mathbf{V}_S\cdot\nabla s_S-\dot s_S=0$

Table 10.1. Summary of properties of some vertical coordinate systems.

many theoretical and numerical studies. Each system has its advantages and disadvantages, which must be weighed with a particular application in mind. At present, there seems to be a movement away from σ or hybrid $\sigma-p$ coordinates and toward θ or hybrid $\sigma-\theta$ coordinates.

10.6 Vertical staggering

After the choice of vertical coordinate system, the next issue is the choice of vertical staggering. Two possibilities are discussed here, and are illustrated in Fig. 10.5. These are the "Lorenz" or "L" staggering, and the "Charney-Phillips" or "CP" staggering. Suppose that both grids have N wind-levels. The L-grid also has N θ-levels, while the CP grid has $N+1$ θ-levels. On both grids, ϕ is hydrostatically determined on the wind-levels, and

$$\phi_l - \phi_{l+1} \sim \theta_{l+\frac{1}{2}}. \tag{125}$$

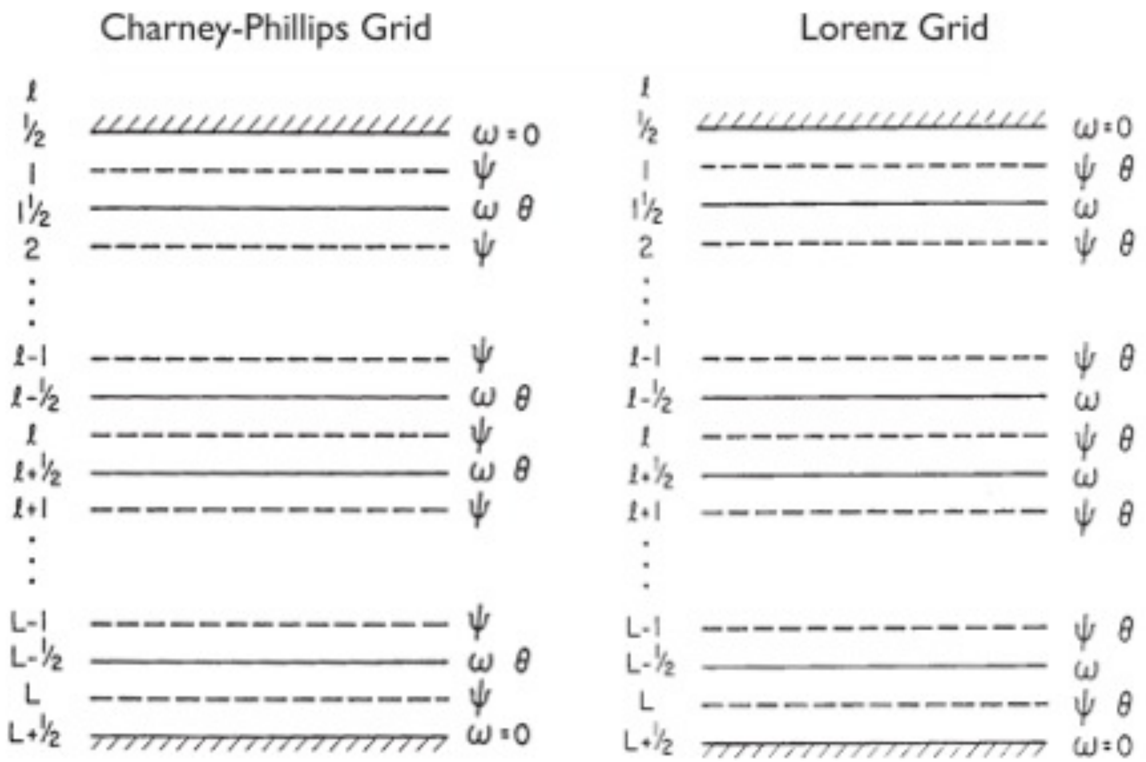

Fig. 10.5: A comparison of the Lorenz and Charney-Phillips staggering methods.

On the CP grid, θ is located between ϕ-levels, so (125) is convenient. With the L-grid, θ must be interpolated. For example, we might choose

$$\phi_l - \phi_{l+1} \sim \frac{1}{2}\left(\theta_l + \theta_{l+1}\right). \tag{126}$$

Because (126) involves averaging, an oscillation in θ is not "felt" by ϕ, and so has no effect on the winds. This allows the possibility of a computational mode in the vertical. No such problem occurs with the CP grid.

There is a second, less obvious problem with the L grid. The vertically discrete potential vorticity corresponding to (109) is

$$q_l \equiv \left(\mathbf{k} \cdot \nabla \times \mathbf{V}_l + f\right)\left(\frac{\partial \theta}{\partial p}\right)_l . \tag{127}$$

It is obvious that (127) "wants" the potential temperature to be defined at levels "in between" the wind levels, as they are on the CP grid. Suppose that we have N wind levels. Then with the CP grid we will have $N+1$ potential temperature levels and N potential vorticities. This is nice. With the L grid, on the other hand, it can be shown that we effectively have $N+1$ potential vorticities. The "extra" degree of freedom in the potential vorticity is spurious, and allows a kind of computational baroclinic instability (Arakawa and Moorthi, 1988). This is a drawback of the L grid.

As Lorenz (1960) pointed out, however, the L-grid is convenient for maintaining total energy conservation, because the kinetic and thermodynamic energies are defined at the same levels. Today, almost all models use the L-grid. This may change.

10.7 Conservation properties of vertically discrete models using σ -coordinates

We now investigate conservation properties of the vertically discretized equations, using σ-coordinates, and *using the L-grid*. The discussion follows Arakawa and Lamb (1977), although some of the ideas originated with Lorenz (1960). For simplicity, we consider only vertical discretization, and keep both the temporal and horizontal derivatives in continuous form.

The following discussion is a bit complicated, so we begin by working out conservation of energy with the continuous equations, first using pressure coordinates, and then using sigma coordinates.

In pressure coordinates, the relevant equations are:

$$\nabla \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0$$

We begin by writing down the prognostic equations of the model. Conservation of mass is expressed, in the vertically discrete system, by

$$\frac{\partial \pi}{\partial t} + \nabla_\sigma \cdot \left(\pi \mathbf{V}_l\right) + \left[\frac{\delta(\pi \dot{\sigma})}{\delta \sigma}\right]_l = 0 , \tag{128}$$

where

$$\left[\delta(\ )\right]_l \equiv (\ )_{l+\frac{1}{2}} - (\ )_{l-\frac{1}{2}} . \tag{129}$$

Similarly, conservation of potential temperature is expressed, in flux form, by

$$\frac{\partial(\pi\theta_l)}{\partial t}+\nabla_\sigma\cdot(\pi\mathbf{V}_l\theta_l)+\left[\frac{\delta(\pi\dot{\sigma}\theta)}{\delta\sigma}\right]_l=0\,. \tag{130}$$

Here we omit the heating term, for simplicity. Finally, the momentum equation is

$$\frac{\partial\mathbf{V}_l}{\partial t}+\left[f+\mathbf{k}\cdot\left(\nabla_\sigma\times\mathbf{V}_l\right)\right]\mathbf{k}\times\mathbf{V}_l+\left(\dot{\sigma}\frac{\partial\mathbf{V}}{\partial\sigma}\right)_l+\nabla K_l=-\nabla\phi_l-\left(\sigma\alpha\right)_l\nabla\pi\,. \tag{131}$$

Here $K\equiv\frac{1}{2}\mathbf{V}\cdot\mathbf{V}$ is the kinetic energy per unit mass, and we omit the friction term, for simplicity. Eqs. (128), (130), and (131) contain various symbols that have not yet been defined. For example, in (131), we have to invent a method to compute the horizontal pressure-gradient force. This and related issues will be discussed below.

To complete the system, we need the upper and lower boundary conditions

$$\dot{\sigma}_{\frac{1}{2}}=\dot{\sigma}_{L+\frac{1}{2}}=0\,. \tag{132}$$

We define the vertical coordinate, σ, at layer edges, which are denoted by half-integer subscripts. The change in across a layer is written as $\delta\sigma_l$. Note that

$$\sum_{l=1}^{L}\delta\sigma_l=1\,, \tag{133}$$

$$p_{l+\frac{1}{2}}=\pi\sigma_{l+\frac{1}{2}}+p_T\,, \tag{134}$$

where p_T is a constant, and the constant values of $\sigma_{1+\frac{1}{2}}$ are assumed to be prescribed for each layer edge. Eq. (134) tells how to compute layer-edge pressures. The method to determine layer-center pressures will be discussed later.

By combining (133) with (128), we obtain

$$\frac{\partial\pi}{\partial t}+\nabla\cdot\left\{\sum_{l=1}^{L}\left[\left(\pi\mathbf{V}_l\right)\left(\delta\sigma_l\right)\right]\right\}=0\,, \tag{135}$$

which is the vertically discrete form of the surface pressure tendency equation. From (135), we see that mass is, in fact, conserved, i.e., the vertical mass fluxes do not produce any net source or sink of mass.

In order to use (130) it is necessary to define values of θ at the layer edges, via an interpolation. In Chapter 4 we discussed the interpolation issue in the context of horizontal advection, and that discussion applies to vertical advection as well. As one possibility, the interpolation methods that allow conservation of an arbitrary function of the advected quantity can be used for vertical advection.

Consider the horizontal pressure-gradient force, in connection with (87) and (88). A finite-difference analog of (88) is

$$\pi\left(\mathbf{HPGF}\right)_l = \left[\frac{\delta\left(\sigma\phi\right)}{\delta\sigma}\right]_l \nabla\pi - \nabla\left(\pi\phi_l\right). \tag{136}$$

Multiplying (136) by $\delta\sigma_l$, and summing over all layers, we obtain

$$\begin{aligned}\sum_{l=1}^{L}\pi\left(\mathbf{HPGF}\right)_l\left(\delta\sigma\right)_l &= \sum_{l=1}^{L}\left[\delta\left(\sigma\phi\right)\right]_l\nabla\pi - \sum_{l=1}^{L}\left[\nabla\left(\pi\phi_l\right)\left(\delta\sigma\right)_l\right] \\ &= \phi_S\nabla\pi - \nabla\left\{\sum_{l=1}^{L}\left[\left(\pi\phi_l\right)\left(\delta\sigma\right)_l\right]\right\}.\end{aligned} \tag{137}$$

This is analogous to Eq. (30), which applies in the continuous system. Inspection of (137) shows that, if we use the form of the HPGF given by (136), the vertically summed HPGF cannot spin up or spin down a circulation inside a closed path, in the absence of topography (Arakawa and Lamb, 1977). A vertical differencing scheme of this type is often said to be “angular-momentum conserving” (e.g., Simmons and Burridge, 1981).

The idea outlined above provides a rational way to choose which of the many possible forms of the HPGF should be used in the model. At this point, of course, the form is not fully determined, because we do not yet have a method to compute either ϕ_l or the layer-edge values of ϕ that appear in (135).

Eq. (136) is equivalent to

$$\pi\left(\mathbf{HPGF}\right)_l = \left\{\left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l - \phi_l\right\}\nabla\pi - \pi\nabla\phi_l. \tag{138}$$

By comparison with (87), we identify

$$\pi(\sigma\alpha)_l = \phi_l - \left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l . \tag{139}$$

The corresponding equation is true in the continuous case. Eq. (139) will be used later.

Next consider total energy conservation. We begin by reviewing the continuous case. Potential temperature conservation is expressed by

$$\frac{\partial(\pi\theta)}{\partial t} + \nabla\cdot(\pi\mathbf{V}\theta) + \frac{\partial}{\partial\sigma}(\pi\dot{\sigma}\theta) = 0 . \tag{140}$$

We have assumed no heating for simplicity. Using continuity, this can be expressed in advective form:

$$\frac{\partial\theta}{\partial t} + \mathbf{V}\cdot\nabla\theta + \dot{\sigma}\frac{\partial\theta}{\partial\sigma} = 0 . \tag{141}$$

With the use of the definition of θ , i.e.,

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa} , \tag{142}$$

and the equation of state, (141) can be used to obtain the thermodynamic energy equation in the form

$$c_p\left(\frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla T + \dot{\sigma}\frac{\partial T}{\partial\sigma}\right) = \omega\alpha . \tag{143}$$

Here

$$\begin{aligned}\omega &= \left(\frac{\partial p}{\partial t}\right)_\sigma + \mathbf{V}\cdot\nabla_\sigma p + \dot{\sigma}\frac{\partial p}{\partial\sigma} \\ &= \sigma\left(\frac{\partial\pi}{\partial t} + \mathbf{V}\cdot\nabla\pi\right) + \pi\dot{\sigma} .\end{aligned} \tag{144}$$

Continuity then allows us to transform (142) to the flux form:

$$\frac{\partial}{\partial t}\left(\pi c_p T\right)+\nabla\cdot\left(\pi \mathbf{V} c_p T\right)+\frac{\partial}{\partial\sigma}\left(\pi\dot{\sigma}c_p T\right)=\pi\omega\alpha \ . \tag{145}$$

We will now derive a finite-difference analog of (145), by starting from the vertically discretized flux form of the potential temperature equation, which is (130). For concreteness, suppose that the model explicitly predicts θ_l by using (130). We adopt the definition

$$\theta_l=\frac{T_l}{\Pi_l}\ , \tag{146}$$

where for convenience we define

$$\Pi_l\equiv\left(\frac{p_l}{p_0}\right)^{\kappa} . \tag{147}$$

Phillips (1974) suggested

$$\Pi_l=\left(\frac{1}{1+\kappa}\right)\left[\frac{\delta(\Pi p)}{\delta p}\right]_l , \tag{148}$$

on the grounds that this form leads to a good simulation of vertical wave propagation. Tokioka (1978) showed that with (148), the finite-difference hydrostatic equation (discussed later) is exact for isentropic atmospheres, i.e., for those in which the potential temperature is uniform with height.

The advective form corresponding to (130) is

$$\pi\left(\frac{\partial\theta_l}{\partial t}+\mathbf{V}_l\cdot\nabla\theta_l\right)+\left[\frac{(\pi\dot{\sigma})_{l+\frac{1}{2}}\left(\theta_{l+\frac{1}{2}}-\theta_l\right)+(\pi\dot{\sigma})_{l-\frac{1}{2}}\left(\theta_l-\theta_{l-\frac{1}{2}}\right)}{(\delta\sigma)_l}\right]=0 . \tag{149}$$

Substitute (146) into (149), to obtain the corresponding prediction equation for T_l:

$$\pi\left(\frac{\partial T_l}{\partial t}+\mathbf{V}_l\cdot\nabla T_l\right)-\pi\theta_l\frac{\partial \Pi_l}{\partial \pi}\left(\frac{\partial \pi}{\partial t}+\mathbf{V}_l\cdot\nabla\pi\right)+\left[\frac{(\pi\dot{\sigma})_{l+\frac{1}{2}}\left(\Pi_l\theta_{l+\frac{1}{2}}-T_l\right)+(\pi\dot{\sigma})_{l-\frac{1}{2}}\left(T_l-\Pi_l\theta_{l-\frac{1}{2}}\right)}{(\delta\sigma)_l}\right]=0\ . \tag{150}$$

The derivative $\frac{\partial \Pi_l}{\partial \pi}$ can be evaluated once we have specified the form of Π_l, e.g., from (148). We now introduce the layer-edge temperatures, i.e., $T_{l+\frac{1}{2}}$ and $T_{l-\frac{1}{2}}$ (although a method to determine them has not yet been specified), and rewrite (150) as

$$\begin{aligned}&\pi\left(\frac{\partial T_l}{\partial t}+\mathbf{V}_l\cdot\nabla T_l\right)+\left[\frac{(\pi\dot{\sigma})_{l+\frac{1}{2}}\left(T_{l+\frac{1}{2}}-T_l\right)+(\pi\dot{\sigma})_{l-\frac{1}{2}}\left(T_l-T_{l-\frac{1}{2}}\right)}{(\delta\sigma)_l}\right]\\&=\pi\theta_l\frac{\partial \Pi_l}{\partial \pi}\left(\frac{\partial \pi}{\partial t}+\mathbf{V}_l\cdot\nabla\pi\right)+\left[\frac{(\pi\dot{\sigma})_{l+\frac{1}{2}}\left(T_{l+\frac{1}{2}}-\Pi_l\theta_{l+\frac{1}{2}}\right)+(\pi\dot{\sigma})_{l-\frac{1}{2}}\left(\Pi_l\theta_{l-\frac{1}{2}}-T_{l-\frac{1}{2}}\right)}{(\delta\sigma)_l}\right].\end{aligned} \tag{151}$$

The layer-edge temperatures can simply be cancelled out in (151) to recover (150). Obviously, the left-hand side of (151) can be rewritten in flux form through the use of the vertically discrete continuity equation:

$$\begin{aligned}&\frac{\partial}{\partial t}(\pi T_l)+\nabla\cdot(\pi\mathbf{V}_l T_l)+\left[\frac{\delta(\pi\sigma T)}{\delta\sigma}\right]_l=\\&\pi\theta_l\frac{\partial \Pi_l}{\partial \pi}\left(\frac{\partial \pi}{\partial t}+\mathbf{V}_l\cdot\nabla\pi\right)+\left[\frac{(\pi\dot{\sigma})_{l+\frac{1}{2}}\left(T_{l+\frac{1}{2}}-\Pi_l\theta_{l+\frac{1}{2}}\right)+(\pi\dot{\sigma})_{l-\frac{1}{2}}\left(\Pi_l\theta_{l-\frac{1}{2}}-T_{l-\frac{1}{2}}\right)}{(\delta\sigma)_l}\right].\end{aligned} \tag{152}$$

By comparison of (145) with (152), we identify

$$\frac{\pi(\omega\alpha)_l}{c_p} = \pi\theta_l \frac{\partial \Pi_l}{\partial \pi}\left(\frac{\partial \pi}{\partial t} + \mathbf{V}_l \cdot \nabla \pi\right) + \left[\frac{(\pi\dot{\sigma})_{l+\frac{1}{2}}\left(T_{l+\frac{1}{2}} - \Pi_l \theta_{l+\frac{1}{2}}\right) + (\pi\dot{\sigma})_{l-\frac{1}{2}}\left(\Pi_l \theta_{l-\frac{1}{2}} - T_{l-\frac{1}{2}}\right)}{(\delta\sigma)_l}\right]. \tag{153}$$

This is a finite-difference analog of the not-so-obvious continuous equation

$$\frac{\pi\omega\alpha}{c_p} = \pi\theta \frac{\partial \Pi}{\partial \pi}\left(\frac{\partial \pi}{\partial t} + \mathbf{V} \cdot \nabla \pi\right) + \frac{\partial(\pi\dot{\sigma}T)}{\partial \sigma} - \Pi \frac{\partial(\pi\dot{\sigma}\theta)}{\partial \sigma}.$$

Returning to the continuous case, we now derive the continuous mechanical energy equation, starting from the continuous momentum equation in the form (131). Dotting (131) with **V** gives the mechanical energy equation in the form

$$\left(\frac{\partial K}{\partial t}\right)_\sigma + \mathbf{V} \cdot \nabla_\sigma K + \dot{\sigma}\frac{\partial K}{\partial \sigma} = -\mathbf{V} \cdot \left(\nabla_\sigma \phi + \sigma\alpha\nabla\pi\right). \tag{154}$$

The corresponding flux form is

$$\left[\frac{\partial(\pi K)}{\partial t}\right]_\sigma + \nabla_\sigma \cdot (\pi \mathbf{V} K) + \frac{\partial(\pi\dot{\sigma}K)}{\partial \sigma} = -\pi\mathbf{V} \cdot \left(\nabla_\sigma \phi + \sigma\alpha\nabla\pi\right). \tag{155}$$

The pressure-work term on the right-hand side of (155) has to be manipulated to facilitate comparison with (145). Begin as follows:

$$\begin{aligned}
-\pi\mathbf{V} \cdot \left(\nabla_\sigma \phi + \sigma\alpha\nabla\pi\right) &= -\nabla_\sigma \cdot (\pi\mathbf{V}\phi) + \phi\nabla_\sigma \cdot (\pi\mathbf{V}) - \pi\sigma\alpha\mathbf{V}\cdot\nabla\pi \\
&= -\nabla_\sigma \cdot (\pi\mathbf{V}\phi) - \phi\left[\frac{\partial \pi}{\partial t} + \frac{\partial(\pi\dot{\sigma})}{\partial \sigma}\right] - \pi\sigma\alpha\mathbf{V}\cdot\nabla\pi \\
&= -\nabla_\sigma \cdot (\pi\mathbf{V}\phi) - \frac{\partial(\pi\dot{\sigma}\phi)}{\partial \sigma} + \pi\dot{\sigma}\frac{\partial \phi}{\partial \sigma} - \phi\frac{\partial \pi}{\partial t} - \pi\sigma\alpha\mathbf{V}\cdot\nabla\pi \\
&= -\nabla_\sigma \cdot (\pi\mathbf{V}\phi) - \frac{\partial(\pi\dot{\sigma}\phi)}{\partial \sigma} - \pi\dot{\sigma}\alpha\pi - \phi\frac{\partial \pi}{\partial t} - \pi\sigma\alpha\mathbf{V}\cdot\nabla\pi\ .
\end{aligned} \tag{156}$$

In the final line of (156) we have used hydrostatics. Referring back to (144), we can write

$$\pi\dot{\sigma}\alpha\pi + \phi\frac{\partial \pi}{\partial t} + \pi\sigma\alpha\mathbf{V}\cdot\nabla\pi = \pi\omega\alpha + \frac{\partial}{\partial \sigma}\left(\phi\sigma\frac{\partial \pi}{\partial t}\right). \tag{157}$$

Substitution of (157) into (156) gives

$$-\pi\mathbf{V}\cdot\left(\nabla_{\sigma}\phi+\sigma\alpha\nabla\pi\right)=-\nabla_{\sigma}\cdot\left(\pi\mathbf{V}\phi\right)-\frac{\partial}{\partial\sigma}\left(\pi\dot{\sigma}\phi+\phi\sigma\frac{\partial\pi}{\partial t}\right)-\pi\omega\alpha\ . \tag{158}$$

Finally, plugging (157) back into (154), and collecting terms, we obtain the mechanical energy equation in the form

$$\left[\frac{\partial}{\partial t}(\pi K)\right]_{\sigma}+\nabla_{\sigma}\cdot\left[\pi\mathbf{V}\left(K+\phi\right)\right]+\frac{\partial}{\partial\sigma}\left[\pi\dot{\sigma}\left(K+\phi\right)+\phi\sigma\frac{\partial\pi}{\partial t}\right]=-\pi\omega\alpha\ . \tag{159}$$

Adding (145) and (159) gives a statement of the conservation of total energy:

$$\left\{\frac{\partial}{\partial t}\left[\pi\left(K+c_pT\right)\right]\right\}_{\sigma}+\nabla_{\sigma}\cdot\left[\mathbf{V}\pi\left(K+\phi+c_pT\right)\right]+\frac{\partial}{\partial\sigma}\left[\pi\dot{\sigma}\left(K+\phi+c_pT\right)+\phi\sigma\frac{\partial\pi}{\partial t}\right]=0\ . \tag{160}$$

Integrating this through the depth of an atmospheric column, we find that

$$\frac{\partial}{\partial t}\left[\int_0^1\pi\left(K+c_pT\right)d\sigma\right]+\nabla\cdot\left[\int_0^1\mathbf{V}\pi\left(K+\phi+c_pT\right)d\sigma\right]+\phi_S\frac{\partial\pi}{\partial t}=0\ , \tag{161}$$

which can also be written as

$$\frac{\partial}{\partial t}\left[\int_0^1\pi\left(K+c_pT+\phi_S\right)d\sigma\right]+\nabla\cdot\left[\int_0^1\mathbf{V}\pi\left(K+\phi+c_pT\right)d\sigma\right]=\pi\frac{\partial\phi_S}{\partial t}\ . \tag{162}$$

The right-hand side of (162) represents the work done on the atmosphere if the lower boundary is moving with time, as in an earthquake.

We now carry out essentially the same derivation using the vertically discrete system. Taking the dot product of $\pi\mathbf{V}_l$ with the HPGF for layer l, we write, closely following (156) - (157),

$$
\begin{aligned}
-\pi\mathbf{V}_l\cdot\left[\nabla\phi_l+(\sigma\alpha)_l\nabla\pi\right] &= -\nabla\cdot(\pi\mathbf{V}_l\phi_l)+\phi_l\nabla\cdot(\pi\mathbf{V}_l)-\pi(\sigma\alpha)_l\mathbf{V}_l\cdot\nabla\pi \\
&= -\nabla\cdot(\pi\mathbf{V}_l\phi_l)-\phi_l\left\{\frac{\partial\pi}{\partial t}+\left[\frac{\delta(\pi\dot{\sigma}\phi)}{\delta\sigma}\right]_l\right\}-\pi(\sigma\alpha)_l\mathbf{V}_l\cdot\nabla\pi \\
&= -\nabla\cdot(\pi\mathbf{V}_l\phi_l)-\left[\frac{\delta(\pi\dot{\sigma}\phi)}{\delta\sigma}\right]_l+\left[\frac{(\pi\sigma)_{l+\frac{1}{2}}\left(\phi_{l+\frac{1}{2}}-\phi_l\right)+(\pi\dot{\sigma})_{l-\frac{1}{2}}\left(\phi_l-\phi_{l-\frac{1}{2}}\right)}{(\delta\sigma)_l}\right] \\
&\quad -\phi_l\frac{\partial\pi}{\partial t}-\pi(\sigma\alpha)_l\mathbf{V}_l\cdot\nabla\pi\ .
\end{aligned}
\tag{163}
$$

Continuing down this path, we construct the terms that we need by adding and subtracting

$$
\begin{aligned}
-\pi\mathbf{V}_l\left[\nabla\phi_l+(\sigma\alpha)_l\nabla\pi\right] &= -\nabla\cdot(\pi\mathbf{V}_l\phi_l)-\left[\frac{\delta(\pi\dot{\sigma}\phi)}{\delta\sigma}\right]_l+\left[\pi(\sigma\alpha)_l-\phi_l\right]\frac{\partial\pi}{\partial t} \\
&\quad -\pi\left\{(\sigma\alpha)_l\left(\frac{\partial\pi}{\partial t}+\mathbf{V}_l\cdot\nabla\pi\right)-\left[\frac{(\pi\dot{\sigma})_{l+\frac{1}{2}}\left(\phi_{l+\frac{1}{2}}-\phi_l\right)+(\pi\dot{\sigma})_{l-\frac{1}{2}}\left(\phi_l-\phi_{l-\frac{1}{2}}\right)}{\pi(\delta\sigma)_l}\right]\right\}.
\end{aligned}
\tag{164}
$$

Using (139) in the form

$$
\pi(\sigma\alpha)_l-\phi_l=-\left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l,
\tag{165}
$$

we can rewrite (164) as

$$
\begin{aligned}
-\pi\mathbf{V}_l\cdot\left[\nabla\phi_l+(\sigma\alpha)_l\nabla\pi\right] &= -\nabla\cdot(\pi\mathbf{V}_l\phi_l)-\left\{\frac{\delta\left[\left(\pi\dot{\sigma}+\sigma\frac{\partial\pi}{\partial t}\right)\phi\right]}{\delta\sigma}\right\}_l \\
-\pi\left\{(\sigma\alpha)_l\left(\frac{\partial\pi}{\partial t}+\mathbf{V}_l\cdot\nabla\pi\right)-\left[\frac{(\pi\dot{\sigma})_{l+\frac{1}{2}}\left(\phi_{l+\frac{1}{2}}-\phi_l\right)+(\pi\dot{\sigma})_{l-\frac{1}{2}}\left(\phi_l-\phi_{l-\frac{1}{2}}\right)}{\pi(\delta\sigma)_l}\right]\right\}&.
\end{aligned}
\tag{166}
$$

By comparing with (158), we infer that

$$\pi(\omega\alpha)_l = \pi(\sigma\alpha)_l\left(\frac{\partial\pi}{\partial t} + \mathbf{V}_l \cdot \nabla\pi\right) - \left[\frac{(\pi\dot{\sigma})_{l+\frac{1}{2}}\left(\phi_{l+\frac{1}{2}} - \phi_l\right) + (\pi\dot{\sigma})_{l-\frac{1}{2}}\left(\phi_l - \phi_{l-\frac{1}{2}}\right)}{(\delta\sigma)_l}\right]. \tag{167}$$

We have now reached the crux of the problem. *To ensure total energy conservation, the form of* $\pi(\omega\alpha)_l$ *given by (167) must match that given by (153).* This can be accomplished by setting:

$$(\sigma\alpha)_l = c_p\theta_l \frac{\partial \Pi_l}{\partial \pi}, \tag{168}$$

$$\phi_l - \phi_{l+\frac{1}{2}} = c_p\left(T_{l+\frac{1}{2}} - \Pi_l\theta_{l+\frac{1}{2}}\right), \tag{169}$$

$$\phi_{l-\frac{1}{2}} - \phi_l = c_p\left(\Pi_l\theta_{l-\frac{1}{2}} - T_{l-\frac{1}{2}}\right). \tag{170}$$

Eq. (168) gives an expression for $(\sigma\alpha)_l$. We already had one, though, in Eq. (139). Requiring that these two formulae agree, we obtain

$$\phi_l - \left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l = c_p\pi\theta_l\frac{\partial\Pi_l}{\partial\pi}. \tag{171}$$

This is a finite-difference form of the hydrostatic equation.

By adding θ_l to both sides, and using Eq. (146), Eqs. (169) - (170) can be rewritten as

$$\left(c_pT_{l+\frac{1}{2}} + \phi_{l+\frac{1}{2}}\right) - \left(c_pT_l + \phi_l\right) = \Pi_l c_p\left(\theta_{l+\frac{1}{2}} - \theta_l\right), \tag{172}$$

and

$$\left(c_p T_l + \phi_l\right) - \left(c_p T_{l-\frac{1}{2}} + \phi_{l-\frac{1}{2}}\right) = \Pi_l c_p \left(\theta_l - \theta_{l-\frac{1}{2}}\right), \tag{173}$$

respectively. These are also finite-difference analogs of the hydrostatic equation, in the form $\frac{\partial M}{\partial \theta} = c_p \Pi$. Add one to each subscript in (173), and add the result to (172). This yields

$$\phi_l - \phi_{l+1} = c_p \left(\Pi_{l+1} - \Pi_l\right) \theta_{l+\frac{1}{2}} . \tag{174}$$

If the forms of Π_l and $\theta_{l+\frac{1}{2}}$ are specified, we can use (174) to integrate the hydrostatic equation upward from level $l+1$ to level l.

In (174), the problem with the L grid becomes apparent. We must determine $\theta_{l+\frac{1}{2}}$ by some form of interpolation, e.g., the arithmetic mean of the neighboring layer-center values of θ. The interpolation will "hide" a vertical zig-zag in θ, if one is present in the solution. A hidden zig-zag cannot influence the pressure-gradient force, so it cannot participate in the model's dynamics. Therefore it cannot propagate, as a physical solution would. It can become a permanent, unwanted feature of the simulated sounding.

The problem is actually worse than it may appear at this point. Although we can use (174) to integrate the hydrostatic equation upward, it is still necessary to determine the starting value, ϕ_L , i.e., the layer-center geopotential for the lowest layer. This can be done by first summing $(\delta\sigma)_l$ times (171) over all layers:

$$\sum_{l=1}^{L} \phi_l (\delta\sigma)_l - \phi_S = \sum_{l=1}^{L} \pi c_p \theta_l \frac{\partial \Pi_l}{\partial \pi} (\delta\sigma)_l . \tag{175}$$

Now we use the mathematical identity

$$\begin{aligned} \sum_{l=1}^{L} \phi_l (\delta\sigma)_l &= \sum_{l=1}^{L} \phi_l \left(\sigma_{l+\frac{1}{2}} - \sigma_{l-\frac{1}{2}}\right) \\ &= \phi_L + \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} \left(\phi_l - \phi_{l+1}\right) . \end{aligned} \tag{176}$$

Substitution into (175), and use of (174), gives

$$\phi_L = \phi_S + \sum_{l=1}^{L} \pi c_p \theta_l \frac{\partial \Pi_l}{\partial \pi} (\delta\sigma)_l - \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} c_p \left(\Pi_{l+1} - \Pi_l\right) \theta_{l+\frac{1}{2}} . \tag{177}$$

Eq. (177) is a bit odd, because it says that the thickness between the Earth's surface and the middle of the lowest model layer depends on all of the values of θ throughout the entire column. An interpretation is that all values of θ are being used to estimate the effective value of θ between the surface and level *L*. Since we start from ϕ_L to determine ϕ_l for $l < L$, all values of θ are being used to determine each value of ϕ_l throughout the entire column. This means that *the hydrostatic equation is very non-local*, i.e., the thickness between each pair of layers is determined through an elaborate interpolation that involves the potential temperature at all model levels. Computational modes can run amok.

To avoid this problem, Arakawa and Suarez (1983) proposed an interpolation for $\theta_{l+\frac{1}{2}}$ in which only θ_L influences the thickness between the surface and the middle of the bottom layer. The starting point is to write local hydrostatic equation in the form

$$\phi_l - \phi_{l+1} = c_p \left(A_{l+\frac{1}{2}} \theta_l + B_{l+\frac{1}{2}} \theta_{l+1} \right), \tag{178}$$

where $A_{l+\frac{1}{2}}$ and $B_{l+\frac{1}{2}}$ are non-dimensional parameters to be determined. Comparing with (174), we see that

$$\left(\Pi_{l+1} - \Pi_l\right) \theta_{l+\frac{1}{2}} = A_{l+\frac{1}{2}} \theta_l + B_{l+\frac{1}{2}} \theta_{l+1} . \tag{179}$$

Eq. (179) essentially determines the form of $\theta_{l+\frac{1}{2}}$. In order that it have the form of an interpolation, we must choose $A_{l+\frac{1}{2}}$ and $B_{l+\frac{1}{2}}$ so that $\dfrac{A_{l+\frac{1}{2}} + B_{l+\frac{1}{2}}}{\left(\Pi_{l+1} - \Pi_l\right)} = 1$.

After substitution from (179), Eq. (177) becomes

$$\phi_L - \phi_S = \sum_{l=1}^{L} c_p \pi \theta_l \frac{\partial \Pi_l}{\partial \pi} (\delta\sigma)_l - \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} c_p \left(A_{l+\frac{1}{2}} \theta_l + B_{l+\frac{1}{2}} \theta_{l+1} \right). \tag{180}$$

Every term on the right-hand-side of (180) involves a layer-center value of θ. To eliminate any dependence of ϕ_L on the values of θ above the lowest layer, we "collect terms" around individual values of θ_l, and force the coefficients to vanish for $l < L$. This gives

$$\pi \frac{\partial \Pi_l}{\partial \pi} (\delta\sigma)_l = \sigma_{l+\frac{1}{2}} A_{l+\frac{1}{2}} + \sigma_{l-\frac{1}{2}} B_{l-\frac{1}{2}} \text{ for } l < L . \tag{181}$$

With the use of (181), (177) reduces to

$$\phi_L - \phi_S = \left[\pi \frac{\partial \Pi_l}{\partial \pi} (\delta\sigma)_L - \sigma_{L-\frac{1}{2}} B_{L-\frac{1}{2}} \right] c_p \theta_L , \tag{182}$$

because the coefficient of each θ_l has been forced to vanish for all $l < L$; only the coefficient of ϕ_L is non-zero.

Particular choices for $A_{l+\frac{1}{2}}$ and $B_{l+\frac{1}{2}}$ are discussed by Arakawa and Suarez (1983). The details are omitted here.

10.8 Summary and conclusions

The problem of representing the vertical structure of the atmosphere in numerical models is receiving a lot of attention at present. Among the most promising of the current approaches are those based on isentropic or quasi-isentropic coordinate systems. Similar methods are being used in ocean models.

At the same time, models are more commonly being extended through the stratosphere and beyond, while vertical resolutions are increasing; the era of hundred-layer models appears to be upon us.

Problems

1. Show that $\dfrac{\partial \phi}{\partial \Pi} = -\theta$.

2. Prove that (85) is consistent with (35).

3. For the hybrid sigma-pressure coordinate of Simmons and Burridge (1981), work out:

   a) The form of the pseudo-density, expressed as a function of the vertical coordinate.

   b) A method to determine the vertical velocity, modeled after the method used with σ -coordinates, as explained in connection with Eq. (85). Write down a "recipe" explaining how you would program the calculation of the vertical velocity.

4. Starting from the continuity equation in height coordinates, Eq. (41), derive the continuity equation in the general ζ -coordinate, Eq. (6). Do not use the hydrostatic equation until the very last step of your derivation.

5. Verify that (139) and (153) are correct in the continuous limit.