How can Kalman Filters handle nonlinearity and non-gaussianity ?

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Issues of nonlinearities in data assimilation

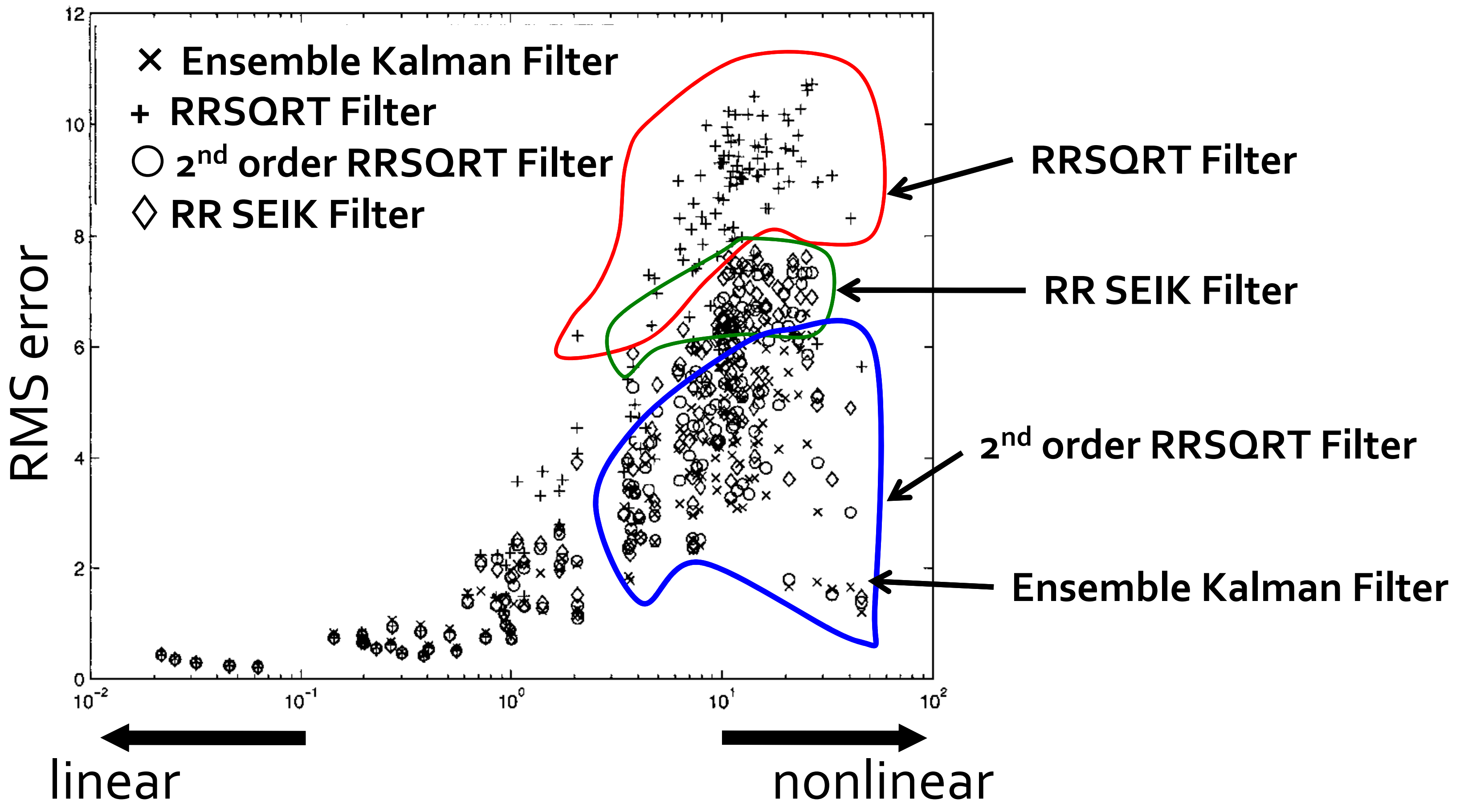
Nonlinearity of the data assimilation problem depends on model dynamics, observations (accuracy, operators, sampling frequency), and the model errors (e.g. Verlaan and Heemink, 2000).

We'll focus on nonlinear dynamics, and propose two new methods for LETKF:

- Outer loop (as in 4D-Var)
- Running in place (for spin-up as well as for long windows)

Verlaan and Heemink (2000):

use the "nonlinearity" to classify the "hardness" of the problem and predict the failure of data assimilation algorithms



With the full nonlinear model, Ensemble Kalman Filter works better than the reduced rank filters for the highly nonlinear cases.

Gaussianity with Ensemble Kalman Filters

With very nonlinear dynamic, deterministic EnKF is more likely to collapse than stochastic EnKF

EnSRF vs. perturbed obs EnKF :

Lawson and Hansen (2004) :

– as nonlinearity becomes appreciable, deterministic filter break down earlier.

Leeuwenburgh et al., (2005):

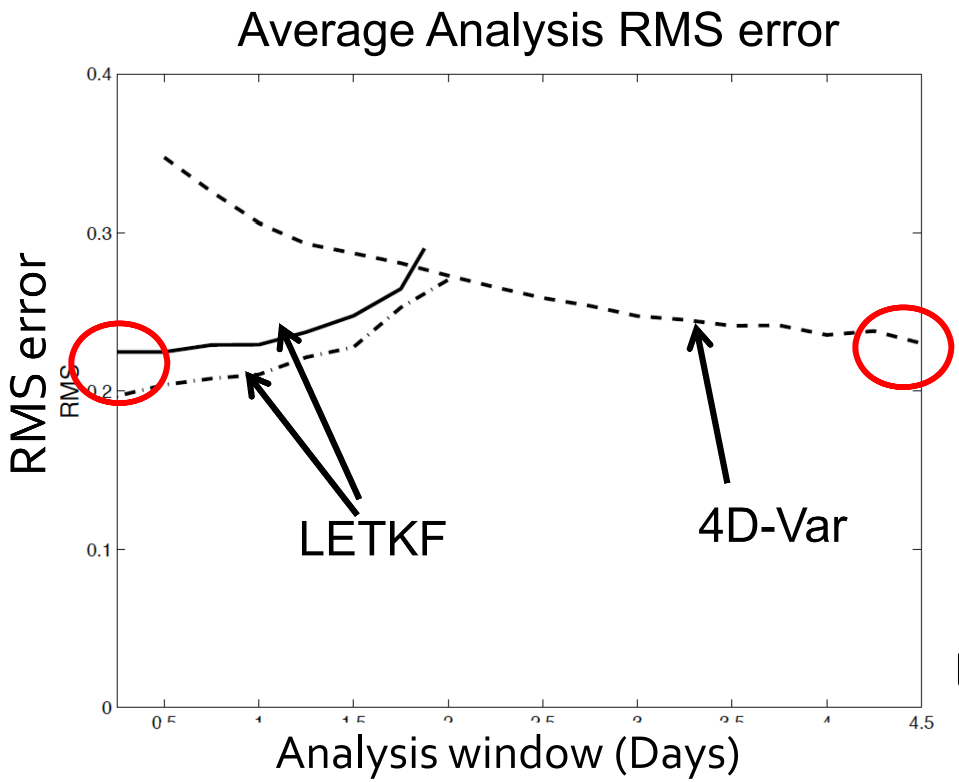
–EnSRF tends to introduce non-Gaussianity. EnSRF+random rotation step helps to improve the Gaussianity.

EnKF vs. 3D/4D-Var

- A disadvantage of ensemble-based KF is that it does not handle nonlinear perturbations well and therefore needs short assimilation windows.
- EnKF doesn't have the important outer loop as in the incremental 3D-Var and 4D-Var (DaSilva, pers. comm. 2006)

- Outer loop is needed to
  - handle the nonlinearities
  - improve the QC process for selecting observations

LETKF v.s. 4D-Var in the Lorenz 40-variable model



Fertig et al., 2007

EnKF does not handle well long windows because ensemble perturbations become non-Gaussian. 4D-Var simply iterates and produces a more accurate control.

Outer-loop in the incremental 4D-Var

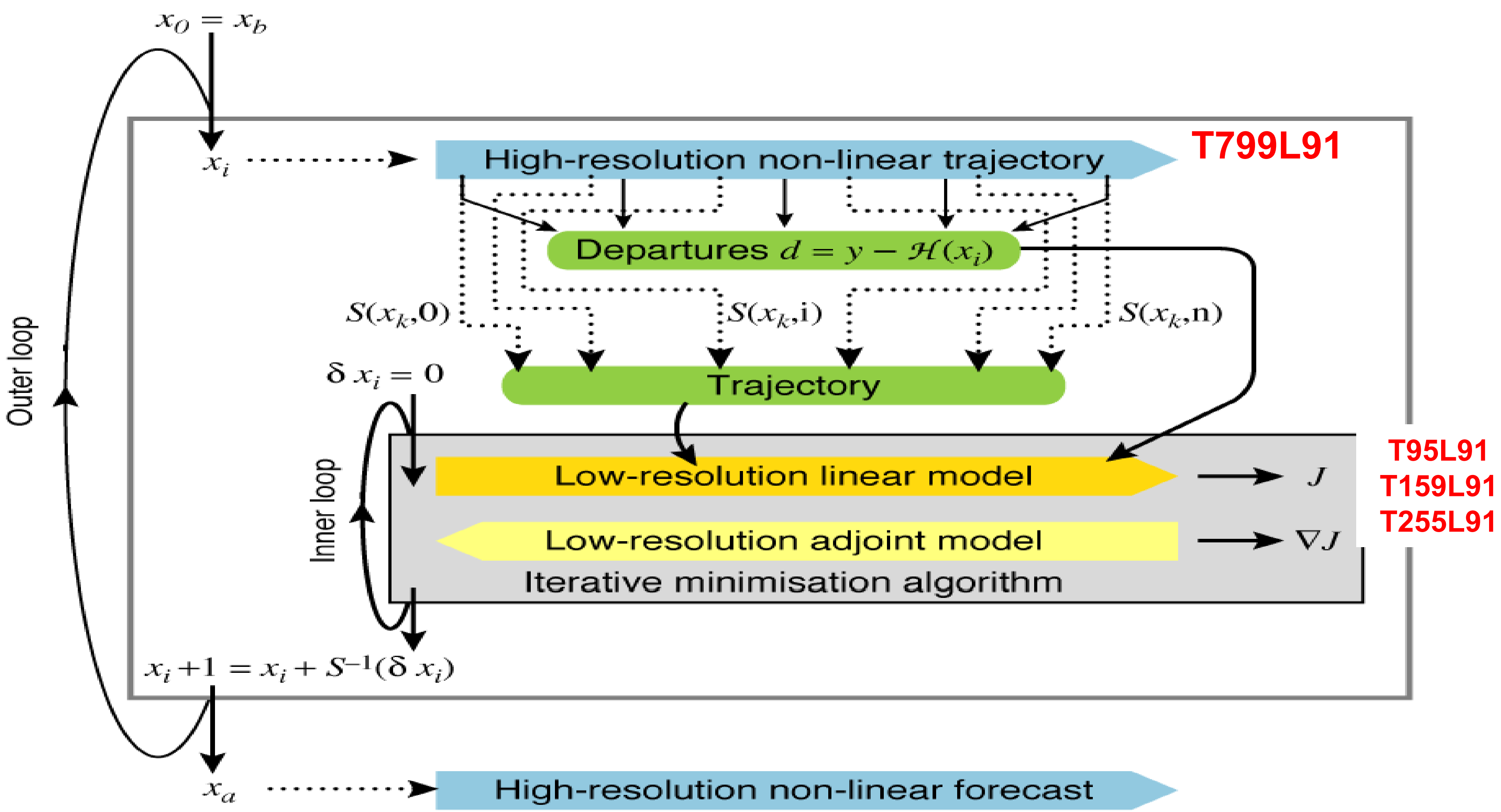
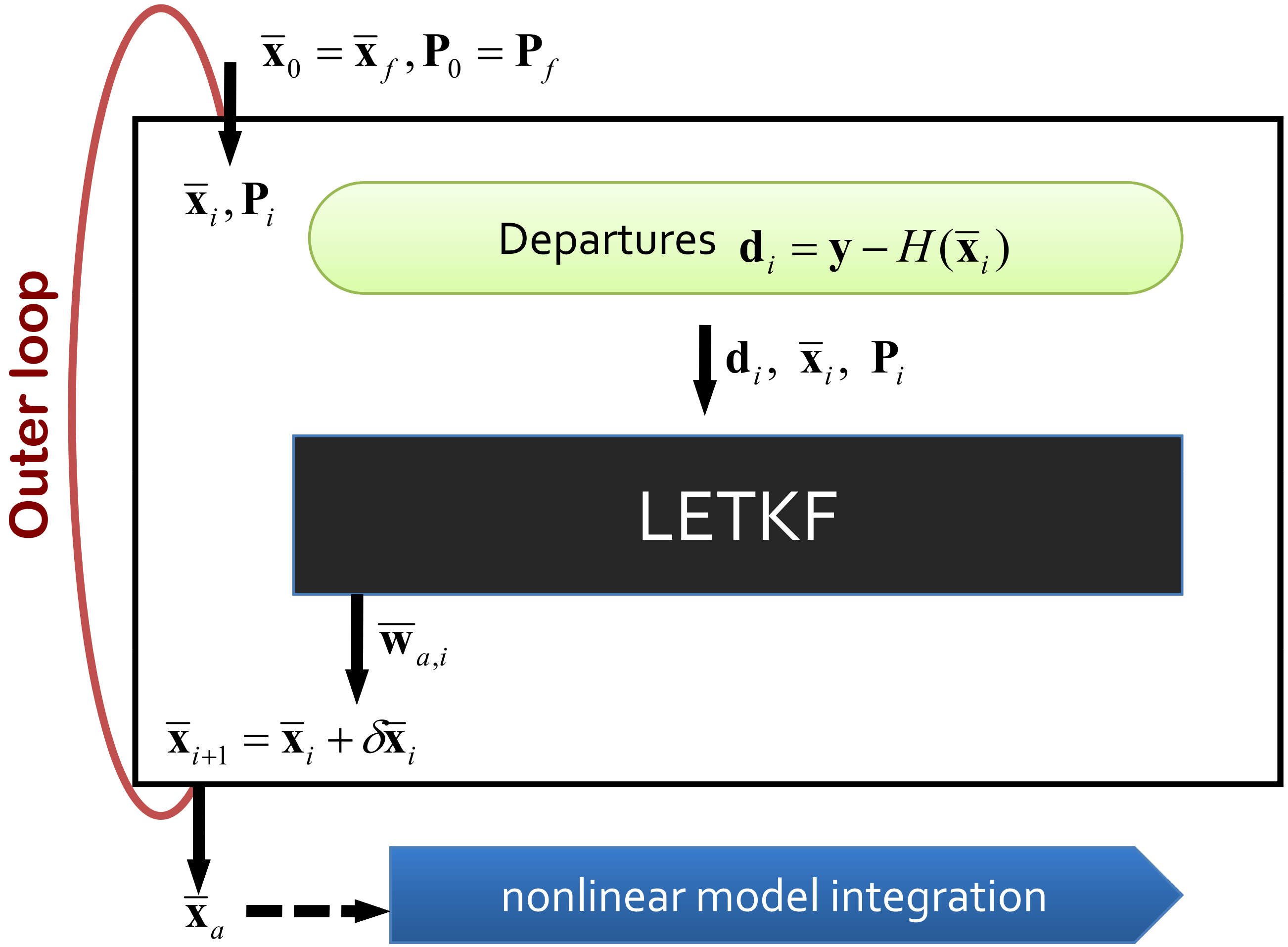


Figure from ECMWF

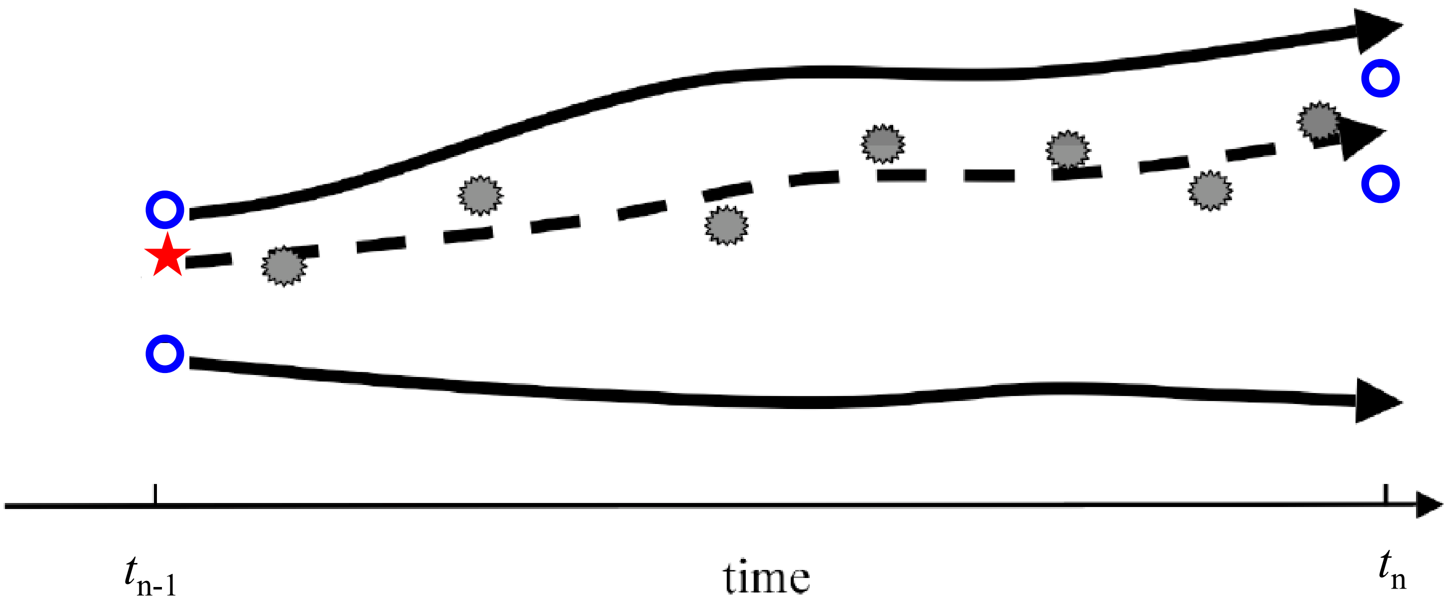
Proposed outer-loop for LETKF

Nonlinear integration in the outer-loop improves the nonlinearity in the background trajectory => adopt the outer-loop for EnKF to improve the **mean trajectory**!



Hunt et al. 2007,
Ott et al., 2004

No-cost smoother for 4D-LETKF
(Kalnay et al, 2007, Yang et al. 2008)



$$\widetilde{\mathbf{P}}_a = [(K-1)\mathbf{I} + \mathbf{Y}_b^T \mathbf{R}^{-1} \mathbf{Y}_b]^{-1};$$

$$\overline{\mathbf{w}}_a = \widetilde{\mathbf{P}}_a \mathbf{Y}_b^T \mathbf{R}^{-1} (\mathbf{y} - H(\overline{\mathbf{x}}));$$

$$\mathbf{W}_a = [(K-1)\widetilde{\mathbf{P}}_a]^{\frac{1}{2}}$$

The LETKF produces an analysis (○) in terms of weights of the ensemble forecast members at the analysis time t_n, giving the trajectory that best fits all the observation in the assimilation window, in 3D or in 4D prospect

No-cost LETKF smoother (★): apply at t_{n-1} the same weights found optimal at t_n, works for 3D- or 4D-LETKF

No-cost LETKF smoother
(Kalnay et al, 2007, Yang et al. 2008)

- LETKF is implemented in the Quasi-Geostrophic channel model
- Observation impact is stored in the ensemble weight coefficients ($\overline{\mathbf{w}}_n$)

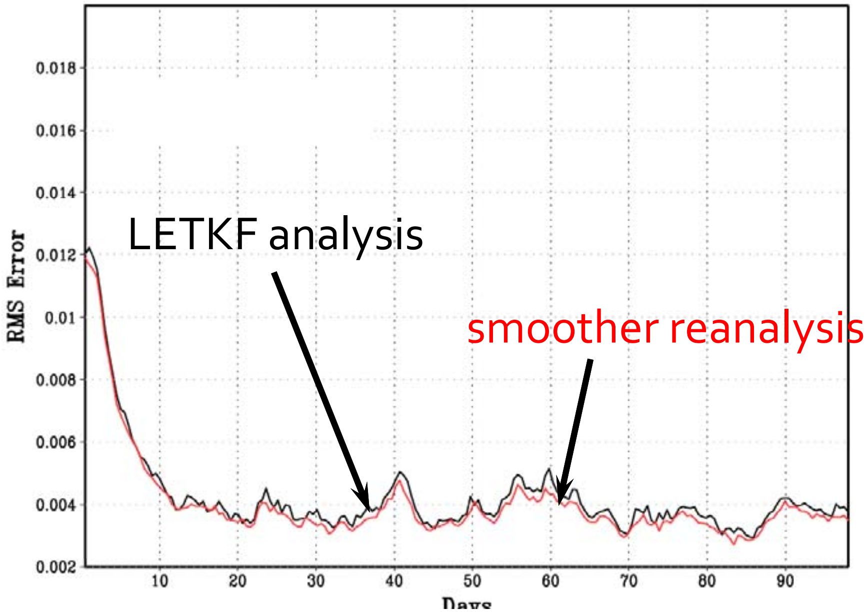
LETKF analysis at time n

$$\overline{\mathbf{x}}_n^a = \overline{\mathbf{x}}_n^f + \mathbf{X}_n^f \overline{\mathbf{w}}_n$$

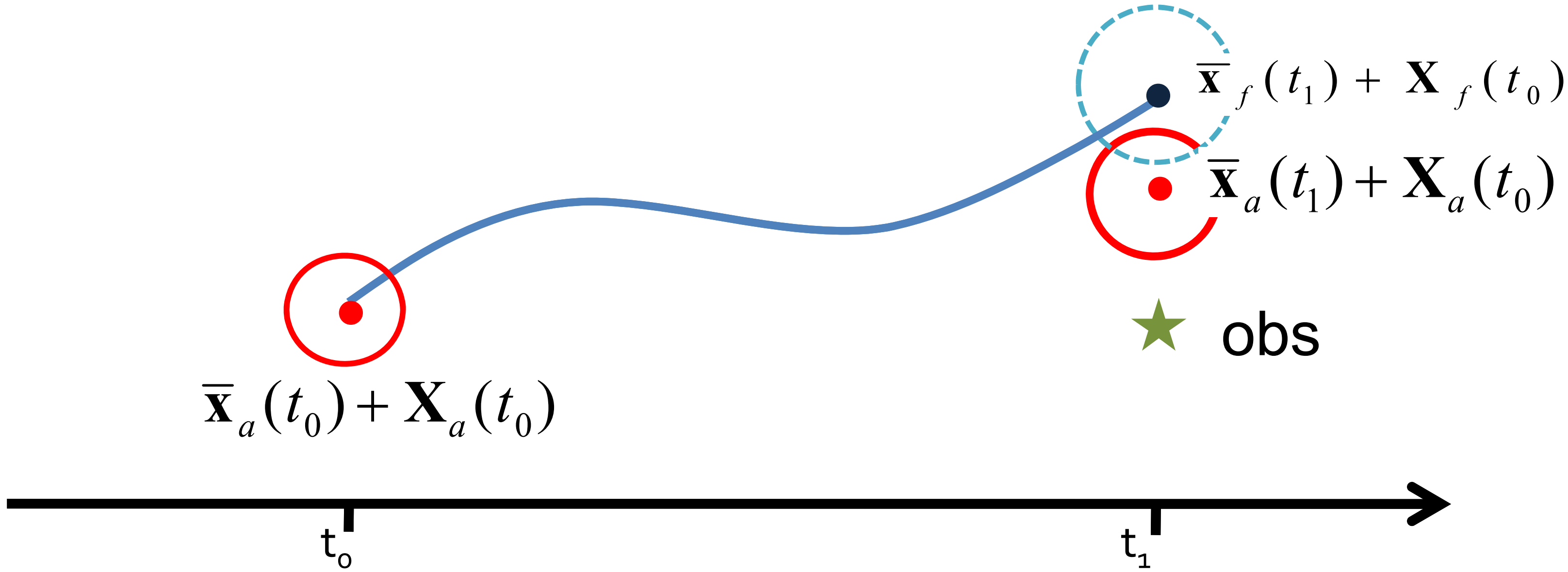
LETKF smoother reanalysis at time n-1

$$\tilde{\overline{\mathbf{x}}}_{n-1}^a = \overline{\mathbf{x}}_{n-1}^a + \mathbf{X}_{n-1}^a \overline{\mathbf{w}}_n$$

RMS analysis error (potential vorticity)



Analysis and ensemble weights from LETKF



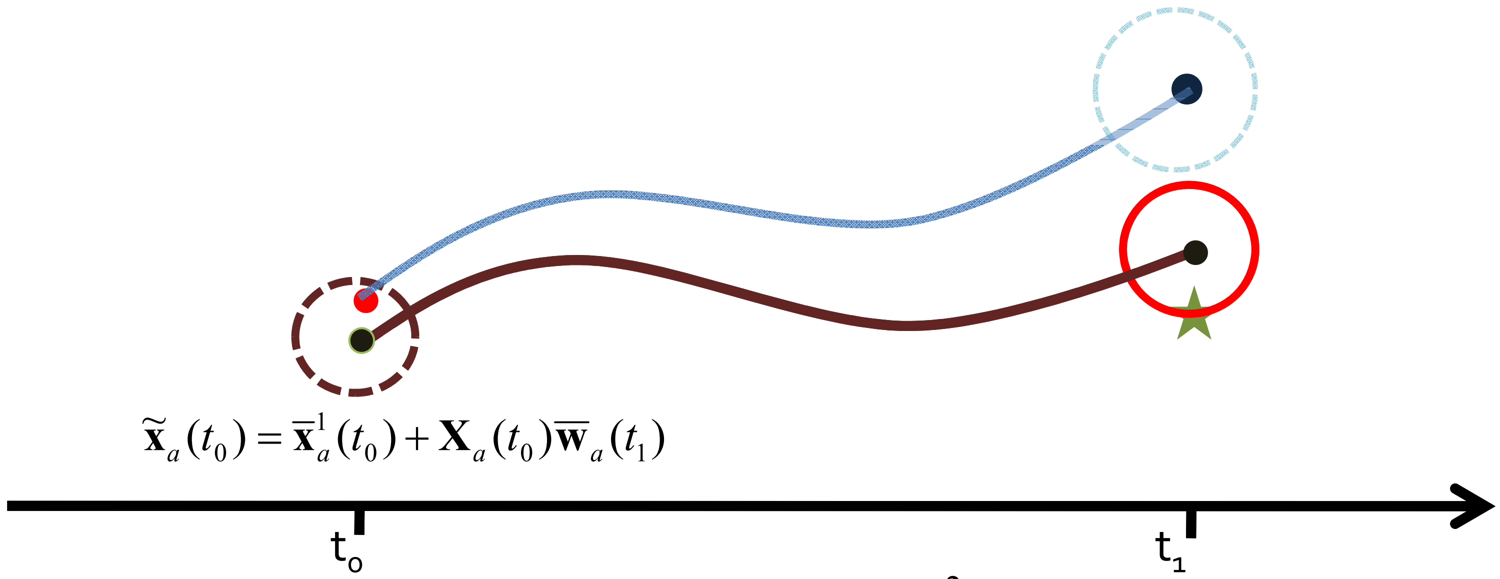
$$\mathbf{X}_a(t_0) = \mathbf{X}_f(t_0)\mathbf{W}_a(t_0)$$
$$\overline{\mathbf{x}}_a(t_1) = \overline{\mathbf{x}}_f(t_1) + \mathbf{X}_f(t_1)\overline{\mathbf{w}}_a(t_1)$$

Outer -loop in LETKF

Outer loop: do the same as 4D-Var, and use the final weights to correct only the mean initial analysis, keeping the initial perturbations. Repeat the analysis once or twice. It centers the ensemble on a more accurate nonlinear solution.

Miyoshi pointed out that Jaszwinski (1970) suggested this in a footnote!!!!!

$$\tilde{\mathbf{x}}_a(t_0) = \overline{\mathbf{x}}_a^1(t_0) + \mathbf{X}_a(t_0)\overline{\mathbf{w}}_a(t_1)$$



$$\overline{\mathbf{x}}_a^{n=2}(t_1) = M(\tilde{\mathbf{x}}_a(t_1)) + \mathbf{X}_a(t_1)\overline{\mathbf{w}}_\mathbf{a}^{n=2}(t_1)$$

Recenter the ensemble
at the improved mean, $M(\tilde{\mathbf{x}}_a(t_1))$

4D-Var vs. LETKF with Lorenz 3-Variable model

Kalnay et al. (2007a, TellusA), RMS analysis error

	4D-Var	LETKF (3 members)
obs every 8 time-steps (linear window)	0.31	0.30
Obs every 25 time-steps (nonlinear window)	0.53 (assim window=75)	0.68 (δ=1.22)

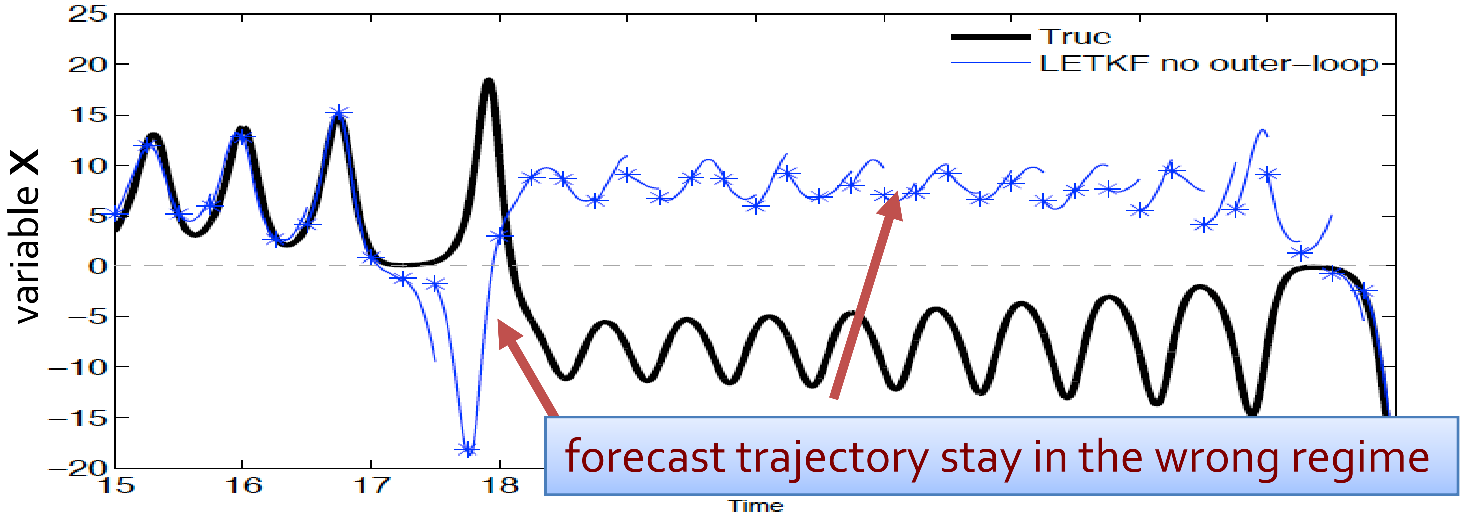
Long window + Pires et al. (1996) -> 4D-Var wins!

With the outer-loop

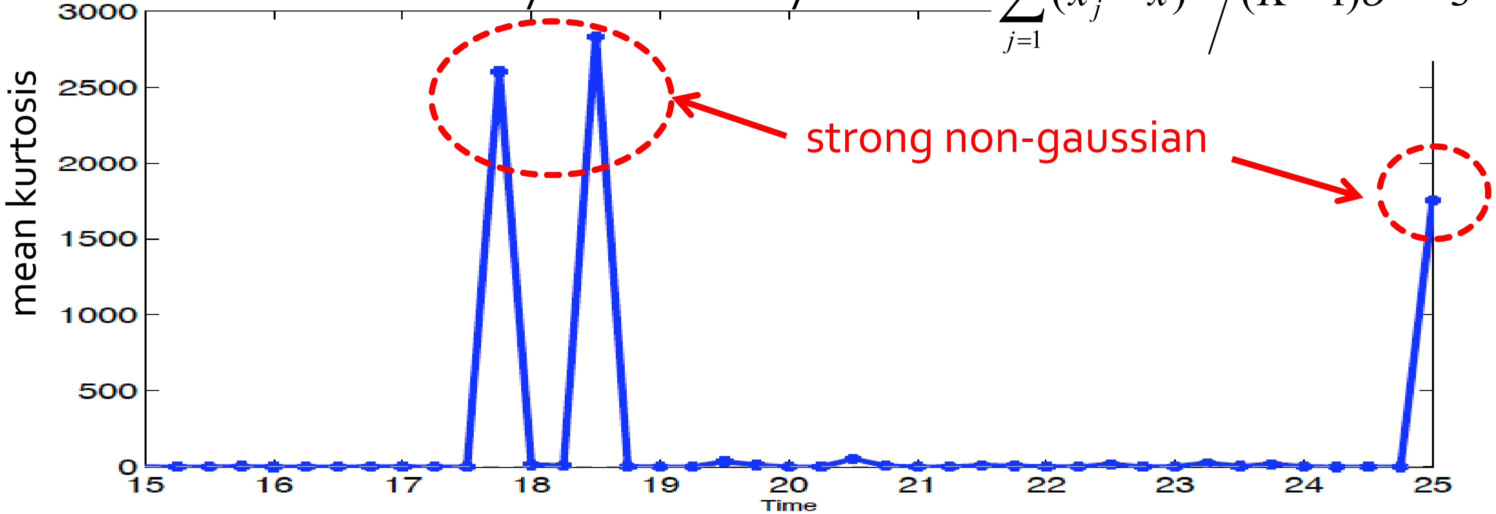
	4D-Var	LETKF (3 ensemble)	LETKF+ outer loop
obs every 8 time-steps (linear window)	0.31	0.30	0.27
Obs every 25 time-steps (nonlinear window)	0.53 (assim window=75)	0.68 (δ=1.22)	0.47 (δ=1.06)

With the outer-loop, LETKF analysis with 25 time-steps is much improved, even better than 4D-Var!

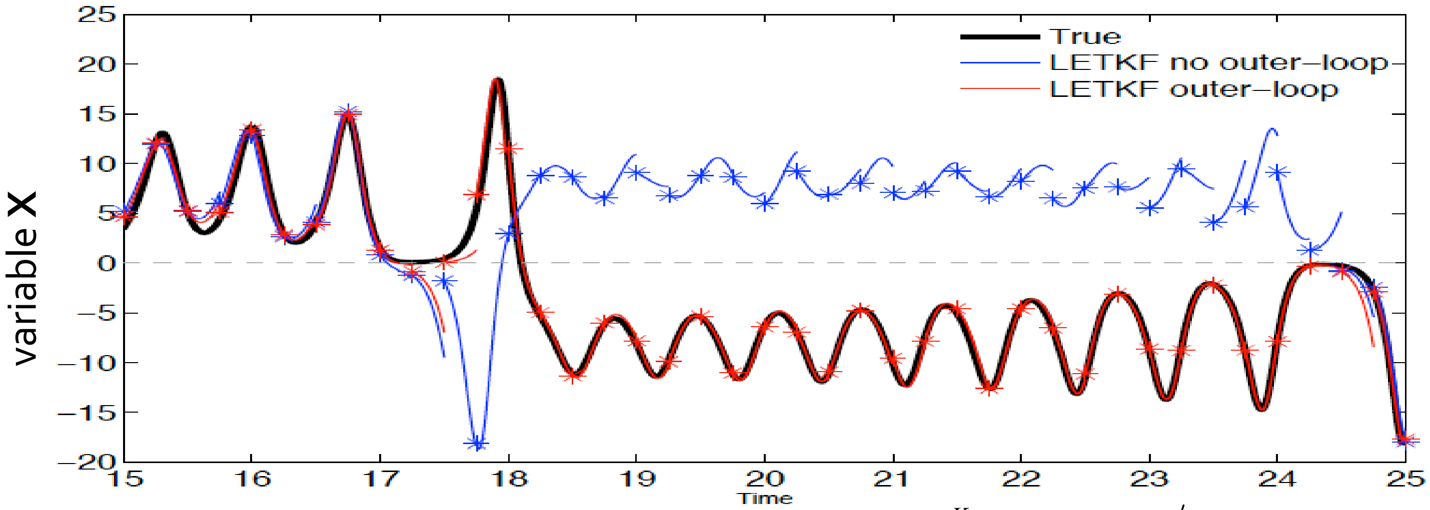
LETKF analysis and Kurtosis



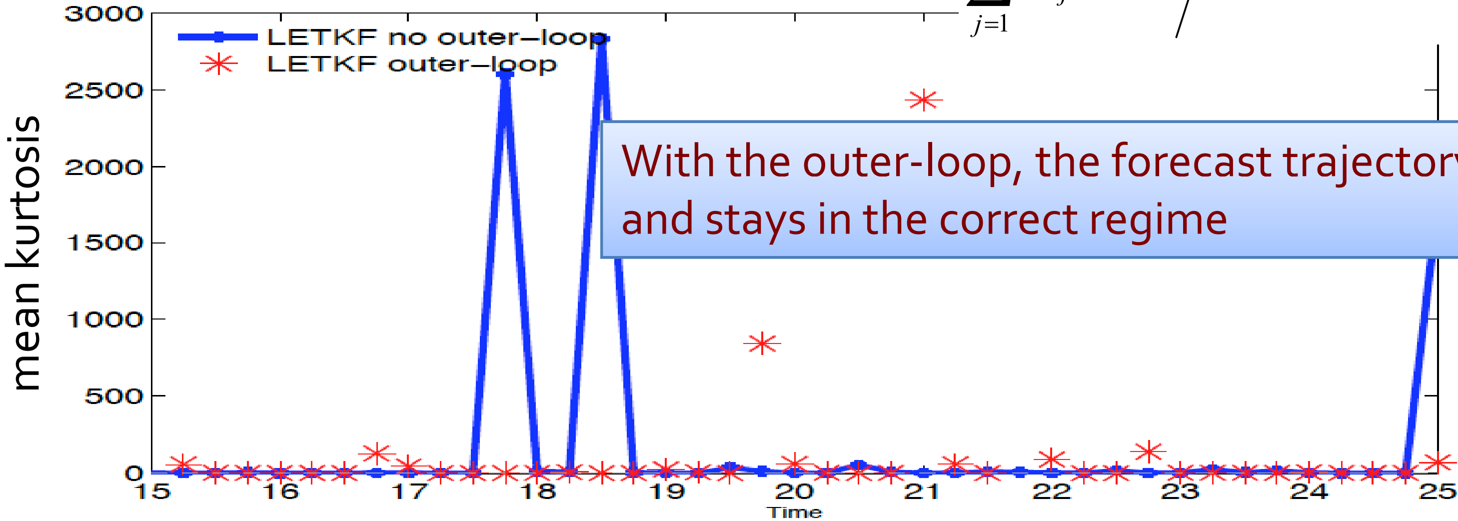
Nonlinearity measured by kurtosis $\sum_{j=1}^{K}(x_j - \bar{x})^4 \Big/ (K-1)\sigma^4 - 3$



LETKF outer-loop analysis and Kurtosis

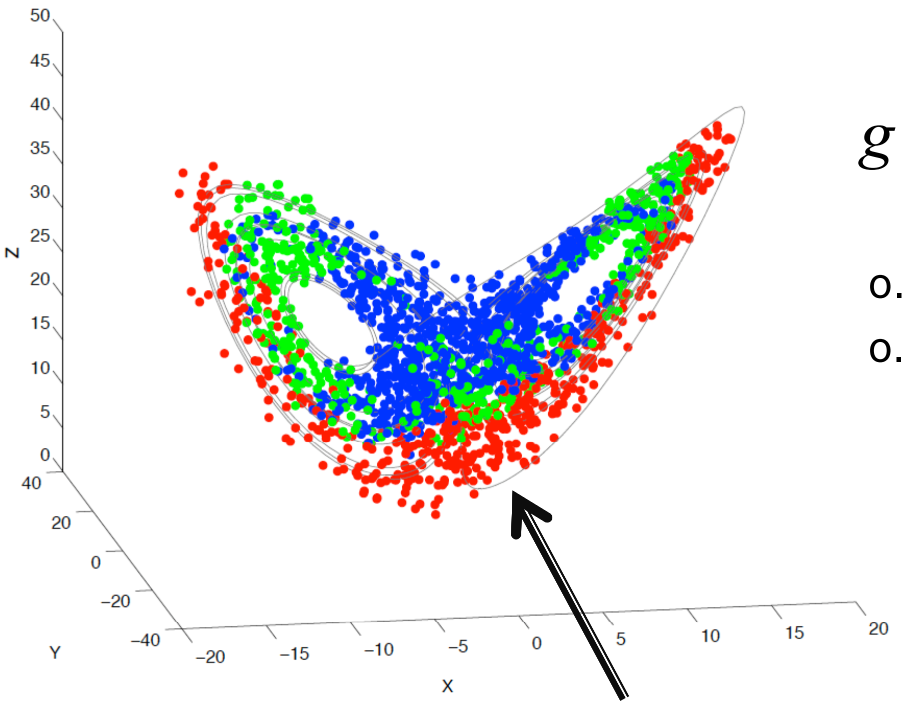


Nonlinearity measured by kurtosis $\sum_{j=1}^{K}(x_j-\bar{x})^4 \Big/ (K-1)\sigma^4 - 3$



With the outer-loop, the forecast trajectory is improved and stays in the correct regime

Impact of outer-loop for regimes of different error growth rate



$$g = \frac{1}{n} \ln\left(\frac{|\delta x|}{|\delta x_0|}\right)$$

0.01 > • > 0.00

0.03 > • > 0.01

• > 0.03

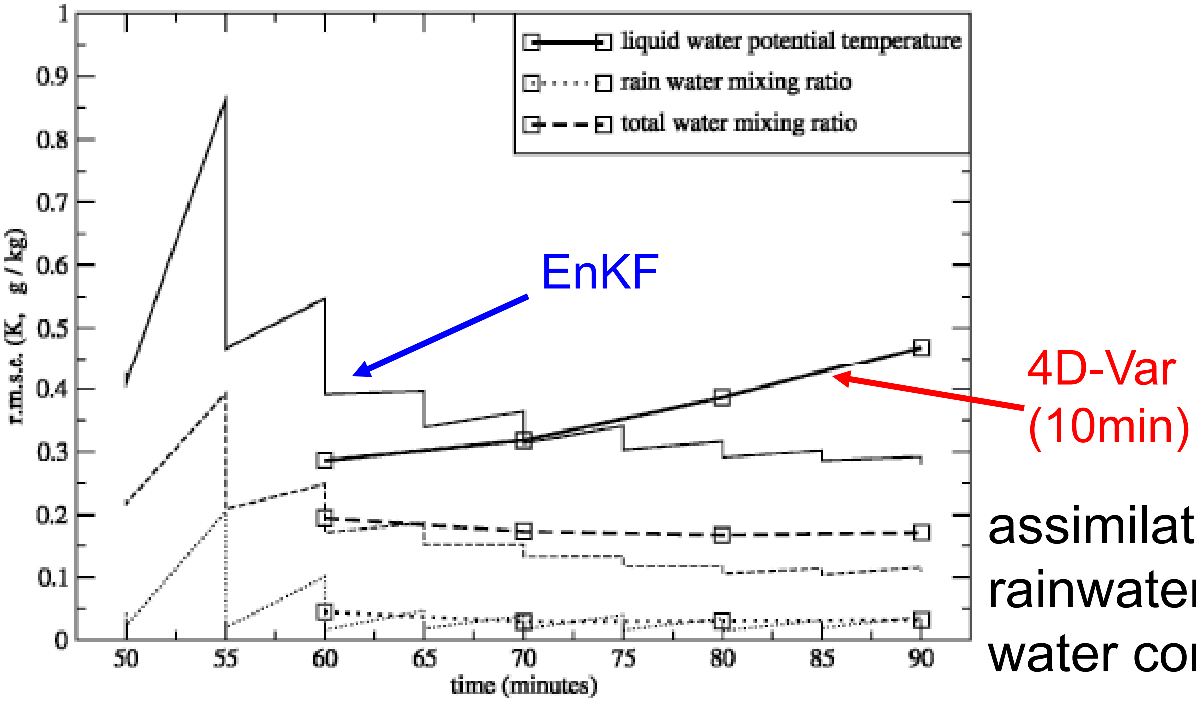
For large error growth rates, the outer-loop is particularly useful to improve the analysis.
No improvement is observed for the linear regimes

Error nonlinearity grows when observations are sparse

- **When the initial ensemble is far away from nature (e.g., cold start), EnKF needs a long spin-up time to reach a satisfactory accuracy.**
  - **Ensemble is less-Gaussian during spin-up**
  - **The accuracy of the mean and the "errors of the day" carried in the perturbations are key factors for good performance of the EnKF.**
  - **EnKF spins up much faster if starting from a good initial condition, e.g. 3D-Var analysis, or perturbations drawn from the B3dvar**
- **4D-Var spins-up faster than EnKF because it is a smoother: it keeps iterating until it fits the observations within the assimilation window as well as possible.**
- **Example: in a severe storm where radar observations start with the storm, there is little real time to spin-up**

Caya et al. (2005) :
EnKF is eventually better than 4D-Var (but it is too late to be useful).

Forecast and analysis RMS error



Time (min)	Observations
20	17
25	150
30	613
35	1114
40	1490
45	1800
50	2063
55	2195
60	2316
65	2521
70	2677
75	2945
80	3186
85	3412
90	3492

assimilate radial velocity, rainwater content and total water content

“Running in Place”

- EnKF is a sequential data assimilation system where, after the new data is used at the analysis time, it should be discarded…

- **only** if the previous analysis and the new background are the most likely states given the past observations.

- **If the system has converged after the initial spin-up all the information from past observations is already included in the background.**

- **During the spin-up we could use the observations repeatedly if we could extract extra information.** But we should avoid overfitting the observations

Running in Place algorithm

a) Perform a standard EnKF analysis and obtain the analysis weights at t_n, saving the mean square observations minus forecast (OMF) computed by the EnKF.

b) Apply the no-cost smoother to obtain the smoothed analysis ensemble at t_{n-1} by using the same weights obtained at t_n.

c) Perturb the smoothed analysis ensemble with a small amount of random Gaussian perturbations, similar to additive inflation.

d) Integrate the perturbed smoothed ensemble to t_n. If the forecast fit to the observations is smaller than in the previous iteration according to some criterion, go to a) and perform another iteration. If not, let $t_{n-1} \leftarrow t_n$ and proceed to the next assimilation window.

Running in Place algorithm (notes)

Notes:

c) *Perturb the smoothed analysis ensemble with a small amount of random Gaussian perturbations, a method similar to additive inflation.*

This perturbation has two purposes:

1) Avoid reaching the same analysis as before, and
2) Encourage the ensemble to explore new unstable directions

d) *Convergence criterion*: if $\frac{OMF^2(iter) - OMF^2(iter+1)}{OMF^2(iter)} > \varepsilon$

with $\varepsilon \sim 5\%$ do another iteration. Otherwise go to the next assimilation window.

Experiments with Lorenz 3-variable model
Outer-loop and Running in Place

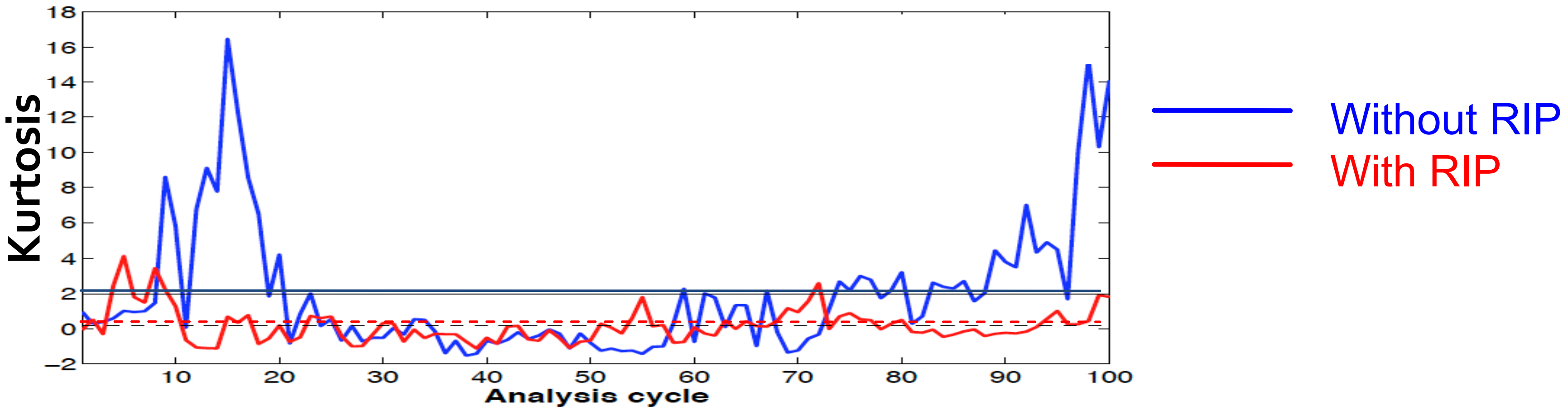
	4D-Var	LETKF (3 ensemble)	LETKF+ outer loop
obs every 8 time-steps (linear window)	0.31	0.30	0.27
Obs every 25 time-steps (nonlinear window)	0.53 (assim window=75)	0.68 (δ=1.22)	0.47 0.37 (RIP)

Running in place gives even more improvement than the outer-loop because it improves both the mean and the covariance.

RIP improves Gaussianity

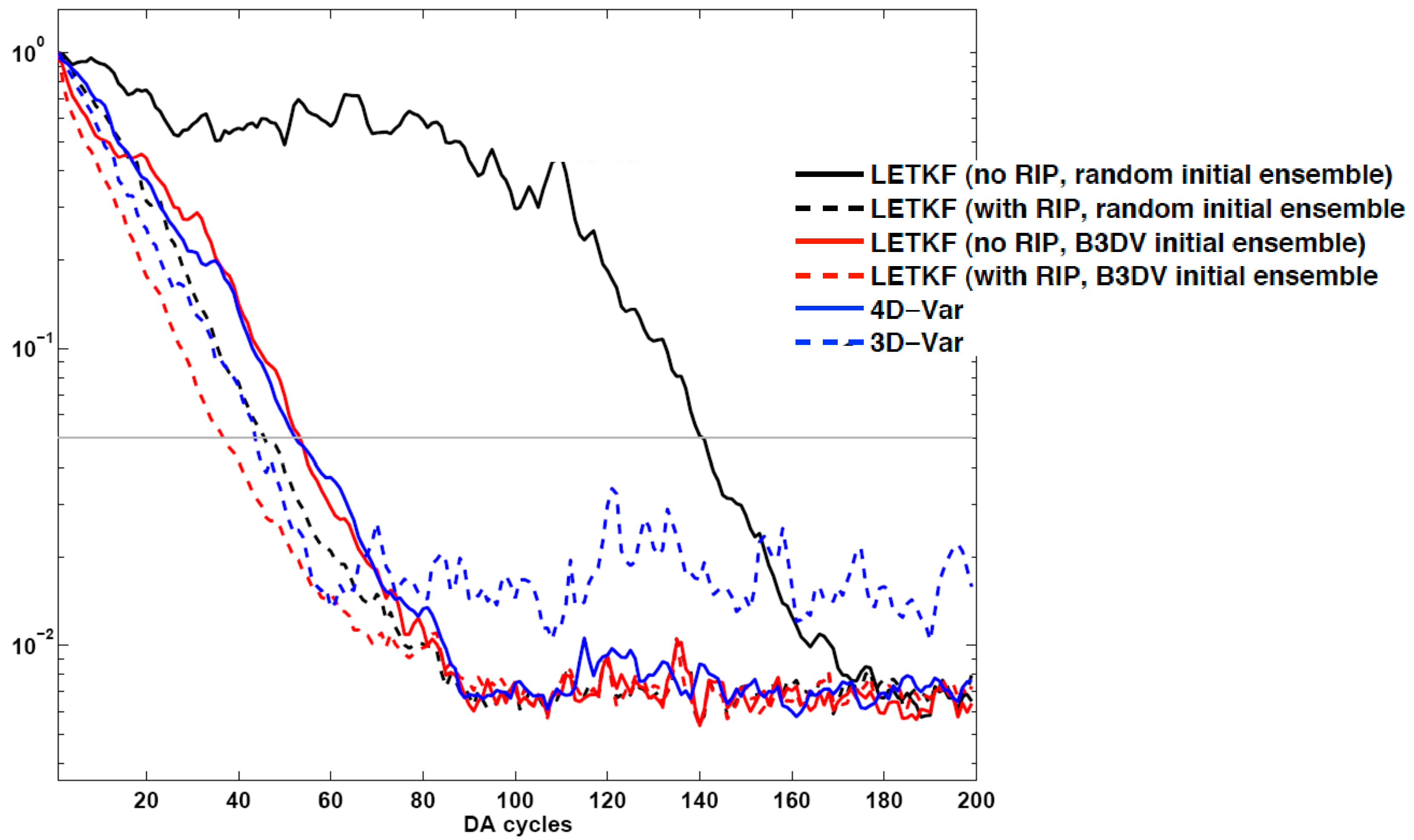
RMS analysis error from LETKF with and without RIP (K: ensemble size)

	Without RIP	With RIP		
	K=24	K=24	K=6	K=3
RMS error (first 10 cycles)	**6.22**	**0.78**	**1.01**	**1.73**
RMS error after converged	**0.55**	**0.34**	**0.35**	**0.37**



- RIP quickly improves the analysis accuracy during the spin-up period and improve the overall analysis accuracy.
- RIP can improve the analysis performance with smaller ensemble size.
- RIP can also improve the Gaussianity of the ensemble perturbations.

RIP LETKF with the Quasi-geostrophic model



RIP LETKF with the Quasi-geostrophic model

	LETKF Random initial ensemble		LETKF B3DV initial ensemble		LETKF Random initial ensemble	Variational	
	No RIP	With RIP	No RIP	With RIP	Fixed 10 iterations RIP	3D-Var B3DV	4D-Var 0.05B3DV
Spin-up: DA cycles to reach 5% error	141	46	54	37	37	44	54
RMS error ($\times 10^{-2}$)	0.5	0.54	0.5	0.52	1.16	1.24	0.54

- LETKF spin-up from random perturbations: 141 cycles. With RIP: 46 cycles
- LETKF spin-up from 3D-Var perts. 54 cycles. With RIP: 37 cycles
- 4D-Var spin-up using 3D-Var prior: 54 cycles.

Summary

- As in the variational methods, an outer-loop with LETKF (EnKF) allows to improve the nonlinear evolution of the background trajectory and better fit the observations.
- “Running in place” improves both the mean (like the outer-loop) and the covariance.
- When the EnKF is initialized from cold start, the “running in place” method helps to achieve a fast spin-up. RIP works well even without any prior information on the statistics.
- During spin-up, the observations can be used more than once if we can extract extra information. The non-Gaussianity in the ensemble can also be reduced during the spin-up.