DSC: Scheduling Parallel Tasks on an Unbounded Number of Processors

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***Abstract*— We present a low-complexity heuristic, named the dominant sequence clustering algorithm (DSC), for scheduling parallel tasks on an unbounded number of completely connected processors. The performance of DSC is, on average, comparable to, or even better than, other higher-complexity algorithms. We assume no task duplication and nonzero communication overhead between processors. Finding the optimum solution for arbitrary directed acyclic task graphs (DAG's) is NP-complete. DSC finds optimal schedules for special classes of DAG's, such as fork, join, coarse-grain trees, and some fine-grain trees. It guarantees a performance within a factor of 2 of the optimum for general coarse-grain DAG's. We compare DSC with three higher-complexity general scheduling algorithms: the ETF by Hwang, Chow, Anger, and Lee; Sarkar's clustering algorithm; and the MD by Wu and Gajski. We also give a sample of important practical applications where DSC has been found useful.**

***Index Terms*— Clustering, directed acyclic graph, heuristic algorithm, optimality, parallel processing, scheduling, task precedence**

I. Introduction

We study the scheduling problem of directed acyclic weighted task graphs (DAG's) on unlimited processor resources and a completely connected interconnection network. The task and edge weights are deterministic. This problem is important in the development of parallel programming tools and compilers for scalable multiple-instruction, multiple-data (MIMD) architectures [17], [19], [21], [23]. Sarkar [17] has proposed a two-step method for scheduling with communication.

1) Schedule an unbounded number of a completely connected architecture. The result of this step will be clusters of tasks, with the constraint that all tasks in a cluster must execute in the same processor.
2) If the number of clusters is larger than the number of processors, then merge the clusters further to the number of physical processors, and also incorporate the network topology in the merging step.

Manuscript received September 8, 1992; revised June 17, 1993. This work was supported in part by the Advanced Research Projects Agency (ARPA) under Contract DABT-63-93-C-0064, in part by the National Science Foundation under Grant DMS-8706122, in part by the the Office of Naval Research under Grant N000149310114, and in part by a startup fund and a Faculty Fellowship from the University of California, Santa Barbara. The content of the information herein does not necessarily reflect the position of the U.S. government, and official endorsement should not be inferred.

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IEEE Log Number 9401222.

In this paper, we present an efficient algorithm for the first step of Sarkar's approach. Algorithms for the second step are discussed elsewhere [23]. The objective of scheduling is to allocate tasks onto the processors and then order their execution so that task dependence is satisfied and the length of the schedule, known as the parallel time, is minimized. In the presence of communication, the complexity of the above scheduling problem has been found to be much more difficult than in the classical scheduling problem, where communication is ignored. The general problem is NP-complete, and even for simple graphs, such as fine-grain trees or the concatenation of a fork and a join, the complexity is still NP-complete [3], [15], and [17]. Only for special classes of DAG's, such as join, fork, and coarse-grain tree, special polynomial algorithms are known [2], [4].

There have been two approaches in the literature addressing the general scheduling problem. The first approach considers heuristics for arbitrary DAG's, and the second studies optimal algorithms for special classes of DAG's. When task duplication is allowed, Papadimitriou and Yannakakis [15] have proposed an approximate algorithm for a DAG with equal task weights and equal edge weights, which guarantees a performance within 50% of the optimum. This algorithm has a complexity of $O(v^3(v \log v + e))$, where v is the number of tasks and e is the number of edges. Kruatrachue and Lewis [13] have also given an $O(v^4)$ algorithm for a general DAG based on task duplication. One difficulty in allowing task duplication is that duplicated tasks may require duplicated data among processors, and thus the space complexity could increase when executing parallel programs on real machines.

Without task duplication, many heuristic scheduling algorithms for arbitrary DAG's have been proposed in the literature (e.g., [12], [17], [19]). A detailed comparison of four heuristic algorithms is given in [9]. One difficulty with most existing algorithms for general DAG's is their high complexity. As far as we know, no scheduling algorithm exists that works well for arbitrary graphs, finds optimal schedules for special DAG's, and also has a low complexity. We present one such algorithm in this paper with a complexity of $O((v + e) \log v)$, called the Dominant Sequence Clustering (DSC) algorithm. We compare DSC with ETF algorithm by Hwang, Chow, Anger, and Lee [11], and discuss its similarities and differences with the MD algorithm proposed by Wu and Gajski [19].

The organization is as follows: Section II introduces the basic concepts. Section III describes an initial design of the DSC algorithm and analyzes its weaknesses. Section IV presents an improved version of DSC that takes care of

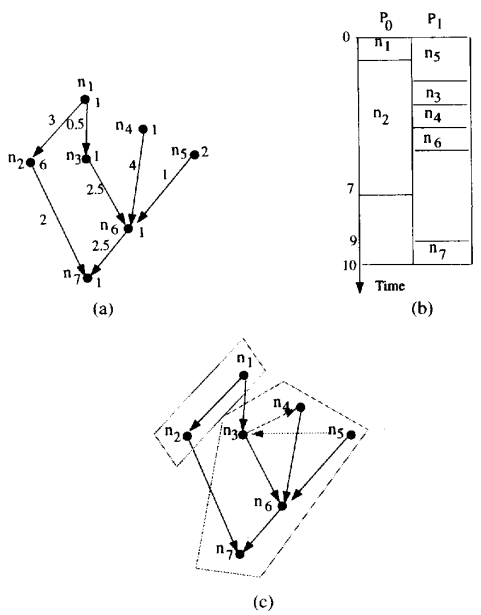

Fig. 1. (a) A weighted DAG. (b) A Gantt chart for a schedule. (c) A scheduled DAG.

the initial weaknesses and analyzes how DSC achieves both low complexity and good performance. Section V gives a performance bound for a general DAG. It shows that the performance of DSC is within 50% of the optimum for coarse-grain DAG's, and it is optimal for join, fork, coarse-grain trees and a class of fine-grain trees. Section VI discusses the related work, presents experimental results and compares the performance of DSC with Sarkar's, ETF, and MD algorithms.

II. Preliminaries

A directed acyclic task graph (DAG) is defined by a tuple $G = (V, E, \mathcal{C}, \mathcal{T})$ where V is the set of task nodes and $v = |V|$ is the number of nodes, E is the set of communication edges, $e = |E|$ is the number of edges, $\mathcal{C}$ is the set of edge communication costs, and $\mathcal{T}$ is the set of node computation costs. The value $c_{i,j} \in \mathcal{C}$ is the communication cost incurred along the edge $e_{i,j} = (n_i, n_j) \in E$, which is zero if both nodes are mapped in the same processor. The value $\tau_i \in \mathcal{T}$ is the execution time of node $n_i \in V$. PRED(n_x) is the set of immediate predecessors of n_x, and SUCC(n_x) is the set of immediate successors of n_x. An example of DAG is shown in Fig. 1(a), with seven tasks $n_1, n_2, \cdots, n_7$. Their execution times are on the right side of the bullets, and edge weights are written on the edges.

The task execution model is the compile time *macro-dataflow* model. A task receives all input before starting execution in parallel, executes to completion without interruption, and immediately sends the output to all successor tasks in parallel. (See Wu and Gajski [19] and Sarkar [17].) Duplication of the same task in separate processors is not allowed.

Given a DAG and an unbounded number of completely connected processors, the scheduling problem consists of two parts: the task to processor assignment, called *clustering* in this paper, and the task execution ordering for each processor. A DAG with a given clustering, but without the task execution ordering, is called a *clustered graph*. A communication edge weight in a clustered graph becomes zero if the start and end nodes of this edge are in the same cluster. Fig. 1(c) shows a clustering of the DAG in Fig. 1(a), excluding the dashed edges for ordering. The tasks n_1 and n_2 constitute one cluster, and the rest of the tasks constitute another. Fig. 1(b) shows a Gantt chart of a scheduling in which the processor assignment for each task and the starting and completion times are defined. An equivalent way of representing a schedule is shown in Fig. 1(c), called a *scheduled* DAG. A scheduled DAG contains both clustering and task execution ordering information. The dashed pseudo edges between n_3 and n_4 in Fig. 1(c) represent the execution ordering imposed by the schedule. The task starting times are not explicitly given in a scheduled DAG, but can be computed by traversing this DAG.

CLUST(n_x) stands for the cluster of node n_x. If a cluster contains only one task, it is called a *unit* cluster. We distinguish between two types of clusters: the linear and the nonlinear. Two tasks are called *independent* if there are no dependence paths between them. A cluster is called *nonlinear* if there are two independent tasks in the same cluster; otherwise, it is called *linear*. In Fig. 1(c), there are two clusters: The cluster that contains n_1 and n_2 is linear; the other cluster is nonlinear, because n_3 and n_4 are independent tasks in Fig. 1(a). A schedule imposes an ordering of tasks in nonlinear clusters. Thus, the nonlinear clusters of a DAG can be thought of as linear clusters in the scheduled DAG if the execution orders between independent tasks are counted as edges. A linear clustering preserves the parallelism present in a DAG, and nonlinear clustering reduces parallelism by sequentializing parallel tasks. A further discussion of this issue can be found in Gerasoulis and Yang [8].

The *critical path* of a clustered graph is the longest path in that graph, including both nonzero communication edge cost and task weights in that path. The parallel time in executing a clustered DAG is determined by the critical path of the scheduled DAG, not by the critical path of the clustered DAG. We call the critical path of a scheduled DAG the *dominant sequence* (DS), to distinguish it from the critical path of the clustered DAG. For Fig. 1(c), the critical path of the clustered graph is $\langle n_1, n_2, n_7 \rangle$, and the DS is still that path. If the weight of n_5 is changed from 2 to 6, then the critical path of the clustered graph remains the same, $\langle n_1, n_2, n_7 \rangle$, but the DS changes to $\langle n_5, n_3, n_4, n_6, n_7 \rangle$. A path that is not a DS is called a SubDS.

Let $tlevel(n_x)$ be the length of the longest path from an entry (top) node to n_x, excluding the weight of n_x in a DAG. Symmetrically, let $blevel(n_x)$ be the length of the longest path from n_x to an exit (bottom) node. For example, in Fig. 1(c), $tlevel(n_1) = 0$, $blevel(n_1) = 10$, $tlevel(n_3) = 2, blevel(n_3) = 4$. The following formula can be used to determine the parallel time from a scheduled graph:

$$\mathrm{PT} = \max_{n_x \in V} \{tlevel(n_x) + blevel(n_x)\}. \quad (1)$$

A. Scheduling as Successive Clustering Refinements

Our approach for solving the scheduling problem with unlimited resources is to consider a scheduling algorithm as performing a sequence of clustering refinement steps. As a matter of fact, most of the existing algorithms can be characterized by using such a framework [9]. The initial step assumes that each node is mapped in a unit cluster. At each step, the algorithm tries to improve on the previous clustering by merging appropriate clusters. A merging operation is performed by zeroing an edge cost connecting two clusters.[1]

Sarkar's Algorithm: We consider Sarkar's algorithm [17, pp. 123–131] as an example of an edge-zeroing clustering refinement algorithm. This algorithm first sorts the e edges of the DAG in a decreasing order of edge weights, and then performs e clustering steps by examining edges from left to right in the sorted list. At each step, it examines one edge and zeros this edge if the parallel time does not increase. Sarkar's algorithm requires the computation of the parallel time at each step, and this problem is also NP-complete. Sarkar uses the following strategy for ordering. Order independent tasks in a cluster by using the highest *blevel* first-priority heuristic, where *blevel* is the value computed in the previous step. The new parallel time is then computed by traversing the scheduled DAG in $O(v + e)$ time. Since there are e steps, the overall complexity is $O(e(e + v))$. Fig. 2 shows the clustering steps of Sarkar's algorithm for the DAG in Fig. 1(a). The sorted edge list with respect to edge weights is $\{(n_4, n_6), (n_1, n_2), (n_3, n_6), (n_6, n_7), (n_2, n_7), (n_1, n_3), (n_5, n_6)\}$. Table I traces the execution of this algorithm, where PT stands for the parallel time and PT_i is the parallel time for executing the clustered graph at the completion of step i. Initially, each task is in a separate cluster, as shown in Fig. 2(a), and the thick path indicates the DS whose length is $PT_0 = 13$. At step 1, edge (n_4, n_6) is examined, and PT remains 13 if this edge is zeroed. Thus, this zeroing is accepted. In step 2, 3, and 4, shown in Fig. 2(c), 2(d), and 2(e), all examined edges are zeroed, because each zeroing does not increase PT. At step 5, edge (n_2, n_7) is examined, and, by zeroing it, the parallel time increases from 10 to 11. Thus, this zeroing is rejected. Similarly, at step 6, zeroing (n_1, n_3) is rejected. At step 7, (n_5, n_6) is zeroed and a pseudo edge from n_5 to n_3 is added, because after step 6, $blevel(n_3) = 3$ and $blevel(n_5) = 5$. Finally, two clusters are produced with PT= 10, shown in Fig. 2(f).

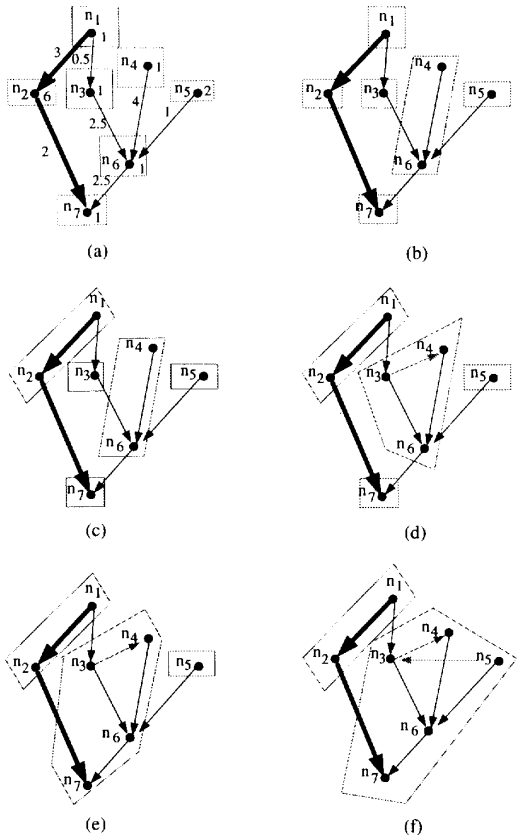

Fig. 2. Clustering steps of Sarkar's algorithm for Fig. 1(a).

TABLE I
CLUSTERING OF STEPS OF SARKAR'S ALGORITHM CORRESPONDING TO FIG. 2

step i	edge examined	PT if zeroed	zeroing	PT_i
0				13
1	(n_4, n_6)	13	yes	13
2	(n_1, n_2)	10	yes	10
3	(n_3, n_6)	10	yes	10
4	(n_6, n_7)	10	yes	10
5	(n_2, n_7)	11	no	10
6	(n_1, n_3)	11	no	10
7	(n_5, n_6)	10	yes	10

III. AN INITIAL DESIGN OF THE DSC ALGORITHM

A. Design Considerations for the DSC Algorithm

As can be seen in the previous section, Sarkar's algorithm zeroes the highest communication edge. This edge, however, may not be in a DS, and as a result, the parallel time may not be reduced at all. (See the zeroing of edge (n_4, n_6) in step 1 Fig. 2.) In order to reduce the parallel time, we must examine the schedule of a clustered graph to identify a DS and then try to reduce its length. *The main idea behind the* DSC *algorithm is to perform a sequence of edge zeroing steps with the goal of reducing the length of a* DS *at each step. The challenge is to implement this idea with low complexity so that it can be used for large task graphs.* Thus, we would like to develop an algorithm having the following goals.

G1: The complexity should be low.

G2: The parallel time should be minimized.

G3: The efficiency should be maximized.

There are several difficulties in the implementation of algorithms that satisfy the above goals.

1) The goals G1, G2, and G3 could conflict with each other. For example, the maximization of the efficiency conflicts with the minimization in the parallel time. When such conflicts arise, G1 is given priority over G2 and G3, and G2 is given priority over G3. In other words, we are

[1] Two clusters will not be merged if there is no edge connecting them, because this cannot decrease the parallel time.

interested in algorithms with low complexity that attain the minimum possible parallel time and the maximum possible efficiency.

2) Let us assume that the DS has been determined. A careful selection of edges to be examined and zeroed must be made to avoid high complexity and simultaneously reduce the parallel time. Consider the example in Fig. 2(a). Initially, the DS is $\langle n_1, n_2, n_7 \rangle$. To reduce the length of that DS, we need to zero at least one edge in DS. Hence, we need to decide which edges should be zeroed. We could zero either one or both edges. If the edge (n_1, n_2) is zeroed, then the parallel time reduces from 13 to 10. If (n_2, n_7) is zeroed, the parallel time reduces to 11. If both edges are zeroed, the parallel time reduces to 9.5. Therefore, there are many possible ways of edge zeroing, and we discuss the following three approaches.

   - **AP1: Multiple DS edge zeroing with maximum PT reduction:** This is a greedy approach that will try to get the maximum reduction of the parallel time at each clustering step. However, it is not always true that multiple DS edge zeroing could lead to the maximum PT reduction, because after zeroing one DS edge of a path, the other edges in that path may not be in the DS of the new graph.
   - **AP2: One DS zeroing of maximum weight edge:** Considering the fact that a DS could become a SubDS by zeroing only one edge of this DS, instead of multiple-edge zeroing, we could zero only one edge at each step to make smaller reductions in the parallel time, but then perform more steps. For example, we could choose one edge to zero at each step, say, the largest weighted edge in DS. In Fig. 2(a), the length of current DS $\langle n_1, n_2, n_7 \rangle$ could be reduced more by zeroing edge (n_1, n_2) instead of edge (n_2, n_7). Zeroing the largest weighted edge may increase the complexity, but may not necessarily lead to a better solution.
   - **AP3: One DS edge zeroing with low complexity:** Instead of zeroing the highest edge weight, we could allow for more flexibility and choose to zero the one DS edge that leads to a low-complexity algorithm. Determining a DS for a clustered DAG could take at least $O(v + e)$ time if the computation is not done incrementally. Repeating this computation for all steps will result in at least $O(v^2)$ complexity. Thus, it is necessary to use an incremental computation of DS from one step to the next to avoid the traversal of the entire DAG at each step. A proper selection of a DS edge for zeroing simplifies the incremental computation of the DS at the next step.

It is not clear how to implement AP1 or AP2 with a low complexity, and also there is no guarantee that AP1 or AP2 will be better than AP3. We use AP3 to develop our algorithm.

1. $EG = \emptyset$. $UEG = V$.
2. Compute *blevel* for each node and set *tlevel* = 0 for each entry node.
3. Every task is marked *unexamined* and assumed to constitute one unit cluster.
4. **While** there is an unexamined node **Do**
5. Find a free node n_f with highest priority from UEG.
6. Merge n_f with the cluster of one of its predecessors such that $tlevel(n_f)$ decreases in a maximum degree. If all zeroings increase $tlevel(n_f)$, n_f remains in a unit cluster.
7. Update the priority values of n_f's successors.
8. $UEG = UEG - \{n_f\}$; $EG = EG + \{n_f\}$.
9. **EndWhile**

Fig. 3. The DSC-I algorithm.

3) Since one of the goals is G3, i.e., reducing the number of unnecessary clusters and increasing the efficiency, we also need to allow for zeroing non-DS edges. The questions is when to do SubDS zeroing. One approach is to always zero DS edges until the algorithm stops, and then follow up with non-DS zeroing. Another approach, followed in this paper, is to interleave the non-DS zeroing with DS zeroing, which gives more flexibility and results in a lower complexity.
4) Since backtracking could result in high complexity, we would like to avoid it as much as possible. For this reason, we need to impose the nonincrease in the parallel time constraint from one step to the next as follows:

$$\mathrm{PT}_{i-1} \geq \mathrm{PT}_i.$$

Sarkar imposes this constraint explicitly in his edge-zeroing process by comparing the parallel time at each step. Here we use an implicit constraint to avoid the explicit computation of parallel time in order to reduce the complexity.

In the next subsection, we present an initial version of DSC algorithm and then identify its weaknesses so that we can improve its performance.

B. DSC-I: An Initial Version of DSC

An algorithmic description of DSC-I is given in Fig. 3. An unexamined node is called *free* if all of its predecessors have been examined. Fig. 4 shows the initial status (step 0), shown in Fig. 4(a), and step i of DSC-I, corresponding to the While loop iteration i at Fig. 3, $1 \leq i \leq 5$, shown in Fig. 4(b)–4(f), in scheduling a DAG with five tasks. The thick paths of each step represent DS's, and the dashed edges represent pseudo execution edges. We provide an explanation of the DSC-I algorithm in Fig. 3.

Priority Definition and DS Identification: **(Line 5 in Fig. 3)** Based on (1), we define the priority for each task at each step as follows:

$$\mathrm{PRIO}(n_f) = tlevel(n_f) + blevel(n_f).$$

Thus, we can identify a DS node as the one with the highest priority. In Fig. 4(a), the DS nodes are n_1, n_2, and n_5, and they have the highest priority value, 14.5.

The Topological Order for Task and Edge Examination: **(Line 5 in Fig. 3)** During the execution of DSC-I, the graph consists of two parts, the examined part, EG, and the unexamined part, UEG. Initially, all nodes are marked *unexamined*,

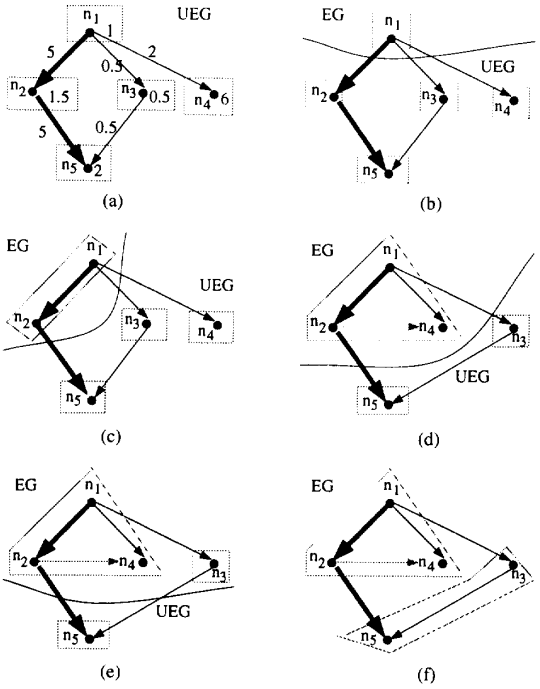

Fig. 4. The result of DSC-I after each step in scheduling a DAG. (a) Step 0. (b) Step 1. (c) Step 2. (d) Step 3. (e) Step 4. (f) Step 5.

(See Fig. 4(a).) DSC-I selects only an unexamined free task n_f with the highest priority. Then it examines an incoming edge of this task for zeroing or not zeroing. In Fig. 4, tasks selected at step 1, 2, 3, 4, and 5 are n_1, n_2, n_4, n_3, and n_5, respectively. Notice that the selected task n_f belongs to a DS if there are free DS nodes; otherwise, it belongs to a SubDS. In Fig. 4, step 1 selects task n_1, which is in a DS; but step 3 selects task n_4, which is not in a DS. The order of such selection is equivalent to *topologically* traversing the graph.

The reason for following a topological order of task examination and zeroing one DS edge is that we can localize the effect of edge zeroing on the priority values for the rest of the unexamined tasks. In this way, even though zeroing changes the priorities of many nodes, DSC-I needs to compute only the changes for nodes that are immediate successors of the currently examined task n_f. (See Line 7 of Fig. 3.) For Fig. 4(b), after n_1 is examined, the priority of n_5 does not need to be updated until one of its predecessors is examined at step 2. More explanations are given by Property 3.2.

Edge-Zeroing Criterion: **(Line 6 in Fig. 3)** The criterion for accepting a zeroing is that the value of $tlevel(n_f)$ of the highest-priority free node *does not increase* by such a zeroing. If it does, then such zeroing is not accepted, and n_f becomes a new cluster in EG. Notice that by reducing $tlevel(n_f)$, all paths going through n_f could be compressed, and as a result, the DS length could be reduced. In Fig. 4(c), n_2 is selected and (n_1, n_2) is zeroed. The zeroing is accepted because $tlevel(n_2)$ reduces from 6 to 1.

Task Placement in EG: **(Line 6 in Fig. 3)** When an edge is zeroed, then a free task n_f is merged to the cluster where one of its predecessors resides. Our scheduling heuristic adds a pseudo edge from the *last* task of this cluster to n_f if they are independent. In Fig. 4(d), n_4 is selected, and the zeroing of (n_1, n_4) is accepted, because $tlevel(n_4)$ reduces from 3 to 2.5. A pseudo edge (n_2, n_4) is added in EG. The DSC-I satisfies the following properties.

Property 3.1: $\mathrm{PT}_{i-1} \geq \mathrm{PT}_i$.

Equation (1) implies that reducing the priority value of tasks would lead to the reduction of the parallel time. Thus, the constraint that *tlevel* values do not increase implies this property. A formal proof is given in Section IV-E.

In DSC-I, *tlevel* and *blevel* values are reused after each step, leading to a reduction of the complexity when determining DS nodes. The following property explains how the topological traversal and the cluster-merging rule of DSC-I reduces the complexity.

Property 3.2: For the DSC-I algorithm, $tlevel(n_x)$ remains unchanged if $n_x \in$ EG and $blevel(n_x)$ remains unchanged if $n_x \in$ UEG.

Proof: If $n_x \in$ UEG, then the topological traversal implies that all descendants of n_x are in UEG. Since n_x and its descendants are in separate unit clusters, $blevel(n_x)$ remains unchanged before it is examined. Also, for nodes in EG, all clusters in EG can be considered "linear" by counting the pseudo execution edges. When a free node is merged with a "linear" cluster, it is always attached to the last node of that "linear" cluster. Thus, $tlevel(n_x)$ remains unchanged after n_x has been examined. □

Property 3.3: The time complexity of the DSC-I algorithm is $O(e + v \log v)$.

Proof: From Property 3.2, the priority of a free node n_f can be easily determined by using

$$tlevel(n_f) = \max_{n_j \in \mathrm{PRED}(n_f)} \{tlevel(n_j) + \tau_j + c_{j,f}\}.$$

Once $tlevel(n_j)$ is computed after its examination at some step, where n_j is the immediate predecessor of n_f, this value is propagated to $tlevel(n_f)$. Afterward $tlevel(n_j)$ remains unchanged and does not affect the value of $tlevel(n_f)$.

At each step of DSC-I, we maintain a priority list FL that contains all free tasks in UEG. This list can be implemented by using a balanced search tree data structure. At the beginning, FL is empty and there is no initial overhead in setting up this data structure. The overhead occurs while maintaining the proper order among tasks after an insertion or deletion of a task. This operation costs $O(\log |\mathrm{FL}|)$, where $|\mathrm{FL}| \leq v$. Because each task in a DAG is inserted to and deleted from FL once during the entire execution of DSC-I, the total complexity of maintaining FL is at the most $2v \log v$. The main computational cost of DSC-I algorithm is spent in the While loop (Line 4 in Fig. 3). The number of steps (iterations) is v. For Line 5, each step costs $O(\log v)$ for finding the head of FL, and v steps cost $O(v \log v)$. For Line 6, each step costs $O(|\mathrm{PRED}(n_f)|)$ when examining the immediate predecessors of task n_f. For the v steps, the cost is $\sum_{n_f \in V} O(|\mathrm{PRED}(n_f)|) = O(e)$. For Line 7, each step costs $O(|\mathrm{SUCC}(n_f)|)$ to update the priority values of the immediate successors of n_f, and, similarly, the cost for v steps is $O(e)$.

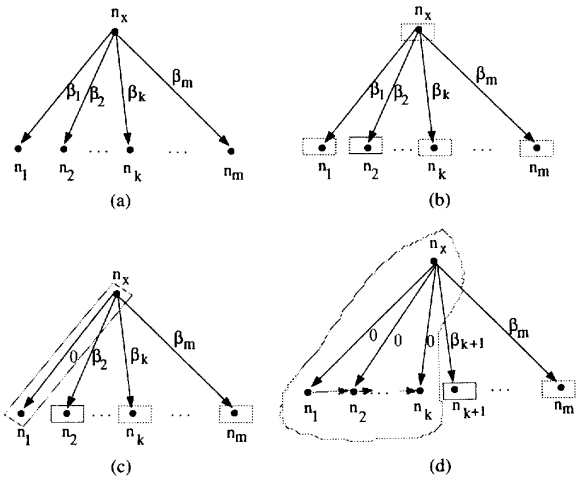

Fig. 5. (a) Fork DAG. (b) Initial clustering. (c) Step 2, $n1$ is examined. (d) Step $k+1$. (c) and (d) The results of DSC-I after step 2 and $k+1$ for a fork DAG.

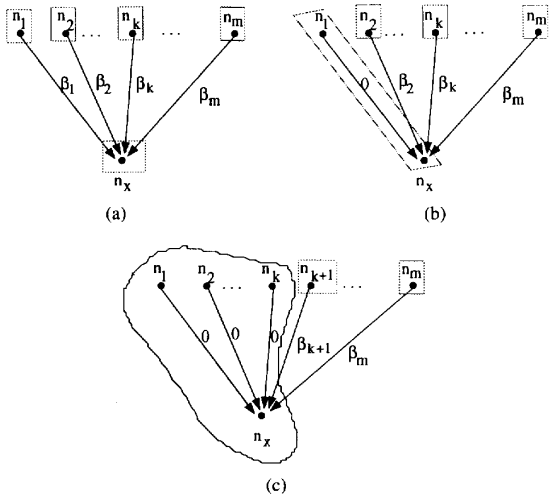

Fig. 6. DSC-I clustering for a join DAG. (a) Join DAG, initial clustering. (b) Final clustering for DSC-I. (c) An optimal clustering.

When a successor of n_f is found free at Line 7, it is added to FL, and the overall cost is $O(v \log v)$. Thus, the total cost for DSC-I is $O(e + v \log v)$. □

C. An Evaluation of DSC-I

In this subsection, we study the performance of DSC-I for some DAG's and propose modifications to improve its performance. Because a DAG is composed of a set of join and fork components, we consider the strengths and weaknesses of DSC-I in scheduling fork and join DAG's. Then we discuss a problem arising when zeroing non-DS edges as a result of the topological ordering of the traversal.

DSC-I for fork DAG's: Fig. 5 shows the clustering steps of DSC-I for a fork DAG. Without loss of generality, assume that the leaf nodes in Fig. 5(a) are sorted such that $\tau_j + \beta_j \geq \tau_{j+1} + \beta_{j+1}$, $j = 1 : m-1$. The steps are described below. Fig. 5(b) shows the initial clustering, where each node is in a unit cluster. EG $= \{\}$, n_x is the only free task in UEG, and PRIO$(n_x) = \tau_x + \beta_1 + \tau_1$. At step 1, n_x is selected, and it has no incoming edges. It remains in a unit cluster, and EG $= \{n_x\}$. After that $n_1, n_2, \cdots, n_m$ become free, and n_1 has the highest priority, PRIO$(n_1) = \tau_x + \beta_1 + \tau_1$, and *tlevel* $(n_1) = \tau_x + \beta_1$. At step 2, shown in Fig. 5(c), n_1 is selected and merged with the cluster of n_x and *tlevel*(n_1) is reduced, in a maximum degree, to τ_x. At step $k+1$, n_k is selected. The original leftmost scheduled cluster in Fig. 5(d) is a "linear" chain $n_x, n_1, \cdots, n_{k-1}$. If attaching n_k to the end of this chain does not increase *tlevel* $(n_k) = \tau_x + \beta_k$, the zeroing of edge (n_x, n_k) is accepted, and the new *tlevel* $(n_k) = \tau_x + \sum_{j=1}^{k-1} \tau_j$. Thus, the condition for accepting or not accepting a zeroing can be expressed as $\sum_{j=1}^{k-1} \tau_j \leq \beta_k$.

It is easy to verify that DSC-I always zeros DS edges at each step for a fork DAG, and the parallel time strictly decreases monotonically. It turns out that the DSC-I algorithm is optimal for this case; a proof is given in Section V.

DSC-I for Join DAG's: Let us consider the DSC-I algorithm for join DAG's. Fig. 6 shows a join DAG, the final clustering, and the optimum clustering. Again, we assume that $\tau_j + \beta_j \geq \tau_{j+1} + \beta_{j+1}, j = 1 : m-1$. The steps of DSC-I are described below. Fig. 6(a) shows the initial clustering. Nodes $n_1, n_2, \cdots, n_m$ are free. One DS is $\langle n_1, n_x \rangle$. Step 1 selects n_1, which is in DS. No incoming edge exists for n_1, and no zeroing is performed. The DS is still $\langle n_1, n_x \rangle$ after step 1. Now n_x becomes *partially free*. A node is partially free if it is in UEG and at least one of its predecessors has been examined, but not all of its predecessors have been examined. Step 2 selects n_2, which is not in DS. No incoming edge exists, and n_2 remains in a unit cluster in EG. Step $m+1$ selects n_x, which is now in the DS, and then (n_1, n_x) is zeroed. The final result of DSC-I is shown in Fig. 6(b), which may not be optimal. The optimal result for a join DAG shown in Fig. 6(c) and the previous optimal solution for a fork are symmetrical.

The join example shows that zeroing only one incoming edge of n_x is not sufficient to attain the optimum without backtracking. In general, when a free node is examined, zeroing multiple incoming edges of this free node instead of zeroing one edge could result in a reduction of *tlevel* in a maximum possible degree. As a consequence, the length of DS or SubDS going through this node could be reduced even more substantially. To achieve such a greedy goal, a low-complexity minimization procedure that zeros *multiple* incoming edges of the selected free node is needed to be incorporated in the DSC algorithm.

Chretienne [3] has proposed an optimal algorithm for a fork and join, which zeros multiple edges. The complexity of his algorithm is $O(m \log B)$, where $B = \min\{\sum_{i=1}^{m} \tau_i, \beta_1 + \tau_1\} + \tau_x$. Al-Mouhamed [1] has also used the idea of zeroing multiple incoming edges of a task to compute a lower bound for scheduling a DAG, using an $O(m^2)$ algorithm; but no feasible schedules that reach the bound are produced by his algorithm. Since we are interested in lower-complexity

algorithms, we use a new optimum algorithm with $O(m \log m)$ complexity.

Dominant Sequence Length Reduction Warranty (DSRW): We describe another problem with DSC-I. When a DS node n_y is partially free, DSC-I suspends the zeroing of its incoming edges and examines the current non-DS free nodes according to a priority-based topological order. Assume that *tlevel*(n_y) could be reduced by δ if such a zeroing were not suspended. We should be able to get at least the same reduction for *tlevel*(n_y) at some future step when DSC-I examines the free node n_y. Then the length of DS going through n_y will also be reduced. However, this is not the case with DSC-I.

Fig. 4(c) shows the result of step 2 of DSC-I after n_2 is merged to CLUST(n_1). The new DS depicted by the thick arrow is $\langle n_1, n_2, n_5 \rangle$, and it goes through partially free node n_5. If (n_2, n_5) were zeroed at step 3, *tlevel* (n_5) would have been decreased by $\delta = 5$. Then the length of the current DS $\langle n_1, n_2, n_5 \rangle$ would also have been reduced by 5. But because of the topological traversal rule, a free task n_4 is selected at step 3, because PRIO$(n_4) = 9 \geq PRIO(n_3) = 4.5$. Then n_4 is merged with CLUST(n_1), because *tlevel*(n_4) can be reduced from 3 to $\tau_1 + \tau_2 = 2.5$. Such zeroing affects the future compression of DS $\langle n_1, n_2, n_5 \rangle$. When n_5 is free at step 5, *tlevel* $(n_5) = \tau_1 + \tau_2 + c_{2,5} = 7.5$, and it is impossible to reduce *tlevel* (n_5) further by moving it to CLUST(n_2). This is because n_5 will have to be linked after n_4, which makes *tlevel* $(n_5) = \tau_1 + \tau_2 + \tau_4 = 8.5$.

To guarantee the effective reduction of *tlevel*(n_y) of a partially free DS node n_y at the future step, we impose a condition called DSRW. The basic idea is to avoid the edge-zeroing operations that affect the reduction of *tlevel* (n_y). The details are described in Section IV-C.

IV. The Final Form of the DSC Algorithm

The main improvements to DSC-I are the minimization procedure for *tlevel* (n_f), maintaining a partially free list (PFL) and imposing the constraint DSRW. The DSC algorithm is described in Fig. 7.

The clustering sequence of Fig. 7 starts from examining the entry tasks of a DAG. We call it *forward clustering*. There is another way to obtain a schedule for a DAG. First, invert this DAG by changing the directions of all precedence edges, obtain a schedule using Fig. 7 for the inverted DAG, and then invert this schedule to produce a schedule with the same length for the original DAG. We call it *backward* clustering. These two approaches may have different performance because of different precedence structures. In fact, in Section V-C, we show that forward clustering cannot obtain the optimum for certain trees, but backward clustering can. The complete version of the DSC algorithm performs both forward and backward clustering and chooses the solution with a shorter schedule. We mainly analyze the properties of DSC with respect to forward clustering. Backward clustering has the same properties, except when dealing with an inverted graph, and it does not increase the order of the algorithm complexity.

```
1.  EG = ∅. UEG = V. Add all free entry nodes to FL.
2.  Compute blevel for each node and set tlevel = 0 for each free node.
3.  WHILE UEG ≠ ∅ DO
4.      n_x = head(FL);/* the free task with the highest priority PRIO. */
5.      n_y = head(PFL); /* the partial free task with the highest priority pPRIO.*/
6.      IF (PRIO(n_x) ≥ pPRIO(n_y)) THEN
7.          Call the minimization procedure to reduce tlevel(n_x).
8.          If no zeroing is accepted, n_x remains in a unit cluster.
9.      ELSE
10.         Call the minimization procedure to reduce tlevel(n_x) under constraint
DSRW.
11.         If no zeroing is accepted, n_x remains in a unit cluster.
12      ENDIF
13.     Update the priorities of n_x's successors and put n_x into EG.
14. ENDWHILE
```

Fig. 7. The DSC algorithm.

A. Priority Lists

At each clustering step, we maintain two node priority lists: a partially free list, PFL, and a free list, FL, both sorted in a descending order of their task priorities. When two tasks have the same priority, we choose the one with the most immediate successors. If there is still a tie, we break it randomly. Function *head*(L) returns the first node in the sorted list L, which is the task with the highest priority. If $L = \{\}$, *head*(L) = NULL, and the priority value is set to 0.

The *tlevel* value of a node is propagated to its successors only after this node has been examined. Thus, the priority value of a partially free node can be updated by using only the *tlevel* from its examined immediate predecessors. Because only part of its predecessors are considered, we define the priority of a partially free task as follows:

$$p\text{PRIO}(n_y) = ptlevel(n_y) + blevel(n_y),$$
$$ptlevel(n_y) = \max_{n_j \in \text{PRED}(n_y) \bigcap EG} \{\text{tlevel}(n_j) + \tau_j + c_{j,y}\}.$$

In general, pPRIO$(n_y) \leq$ PRIO(n_y), and if a DS goes through an edge (n_j, n_y), where n_j is an examined predecessor of n_y, then we have pPRIO$(n_y) =$ PRIO$(n_j) =$ PRIO(n_y). By maintaining pPRIO instead of PRIO, the complexity is reduced considerably. As we prove later, maintaining pPRIO does not adversely affect the performance of DSC, because it can still correctly identify the DS at each step.

B. The Minimization Procedure for Zeroing Multiple Incoming Edges

To reduce *tlevel* (n_x) in DSC, a minimization procedure that zeros multiple incoming edges of free task n_x is needed. An optimal algorithm for a join DAG has been described in [8], and an optimal solution is shown in Fig. 6(c). The basic procedure is to first sort the nodes such that $\tau_j + \beta_j \geq \tau_{j+1} + \beta_{j+1}, j = 1 : m-1$, and then to zero edges from left to right, as long as the parallel time reduces after each zeroing, i.e., *linear searching* of the optimal point. This is equivalent to satisfying the condition $(\sum_{j=1}^{k-1} \tau_j \leq \beta_k)$ for each accepted zeroing. Another optimum algorithm for join is to determine the optimum point k, first by using a *binary search* between 1 and m such that $(\sum_{j=1}^{k-1} \tau_j \leq \beta_k)$, and then zero all edges to the left of k.

We modify the optimum binary search join algorithm to minimize *tlevel* (n_x) when n_x is free. The procedure is shown in Fig. 8. Assume that PRED$(n_x) = \{n_1, n_2, \cdots, n_m\}$. Notice

1. Sort the predecessors of n_x such that

$$tlevel(n_j) + \tau_j + c_{j,x} \geq tlevel(n_{j+1}) + \tau_{j+1} + c_{j+1,x}, \quad j = 1 : m-1.$$

2. Let h be the maximum integer from 2 to m such that for $2 \leq t \leq h$ node n_t satisfies the following constraint:

If n_t is not in $CLUST(n_1)$, then n_t does not have any children other than n_x.

3. Find the optimum point k between 1 and h using the binary search algorithm. Zero (n_1, n_x) up to (n_k, n_x) so that $tlevel(n_x)$ is minimum.

Fig. 8. The minimization procedure for DSC.

that all predecessors are in EG, but now CLUST$(n_1), \cdots,$ CLUST(n_m) may not be unit clusters. We sort the predecessors of n_x such that *tlevel* $(n_j) + \tau_j + c_{j,x} \geq tlevel(n_{j+1}) + \tau_{j+1} + c_{j+1,x}$. To reduce *tlevel* (n_x), we must zero the edge (n_1, n_x). A problem arises when the algorithm zeros an edge of a predecessor n_p, which has other children than n_x. When task n_p is moved from CLUST(n_p) to *CLUST*(n_1), the *tlevel* of the children of n_p will be affected. As a result, the length of paths going through those children will most likely increase. The constraint $PT_{i-1} \geq PT_i$ may no longer hold, and maintaining task priorities becomes complicated. To avoid such cases, we exclude from the minimization procedure predecessors[2] that have children other than n_x, unless they are already in CLUST(n_1). The binary search algorithm in Fig. 8 finds the best stopping point k, and those predecessors n_j of n_x $(2 \leq j \leq k)$ that are not in CLUST(n_1) must be extracted from their corresponding clusters and attached to CLUST(n_1). In computing the new *tlevel* (n_x), those predecessors of n_x in CLUST(n_1) are ordered for execution in an increasing order of their *tlevel* values. The *tlevel* ordering could be performed before the binary searching.

We determine the complexity of this procedure as follows. The *tlevel* ordering of all predecessors is done once at a cost of $O(m \log m)$. The ordering for step 1 at Fig. 8 costs $O(m \log m)$. The binary search computes the effect of the ordered predecessors to $tlevel(n_x)$ at a cost of $O(m)$ at each step. The total number of search steps is $O(\log m)$. Thus, the total cost of the above algorithm is $O(m \log m)$. If linear search were used instead, the total cost would increase to $O(m^2)$, because the number of search steps would increase to m.

C. Imposing Constraint DSRW

As we saw previously, when there is no DS going through any free task and there is one DS passing through a partially free node n_y, then zeroing non-DS incoming edges of free nodes can affect the reduction of *tlevel* (n_y) in the future steps. We impose the following constraint on DSC to avoid such side effects.

DSRW: Zeroing incoming edges of a free node should not affect the reduction of $ptlevel(n_y)$ if it is reducible by zeroing an incoming DS edge of n_y.

There are two problems that we must address in the implementation of DSRW. First, we must detect whether $ptlevel(n_y)$ is reducible. Second, we must make sure that DSRW is satisfied.

1) To detect the reducibility of $ptlevel(n_y)$, we must examine the result of the zeroing of an incoming DS edge of n_y. To find such an incoming DS edge, we examine only the result of the zeroing each incoming edge (n_j, n_y) where n_j is a predecessor of n_y and $n_j \in$ EG. As we prove in Section IV-E, $ptlevel(n_y) = tlevel(n_y)$. This implies that such partial reducibility is sufficient to guarantee that if the parallel time was reducible by zeroing the DS incoming edges of a partially free DS node n_y, then *tlevel* (n_y) is reducible when n_y becomes free. Hence, the DS can be compressed at that time.
2) After detecting the partial reducibility at step i for node n_y, we implement the constraint DSRW as follows. Assume that n_p is one examined predecessor of n_y, and that zeroing (n_p, n_y) would reduce $ptlevel(n_y)$. Then no other nodes are allowed to move to CLUST(n_p) until n_y becomes free.

For the example in Fig. 4(c), $n_y = n_5$, $ptlevel(n_5) = \tau_1 + \tau_2 + c_{2,5} = 7.5$, $pPRIO(n_5) = 9.5$, PRIO$(n_3) = 4.5$, and PRIO$(n_4) = 9$. We have the condition that $pPRIO(n_5) >$ PRIO(n_4), which implies, by Theorem 4.1 in Section IV-E, that DS goes through partially free node n_5. In addition, $ptlevel(n_5)$ could be reduced if (n_2, n_5) were zeroed. Then CLUST(n_2) cannot be touched before n_5 becomes free. Thus, n_3 and n_4 cannot be moved to CLUST(n_2), and they remain in the unit clusters in EG. When n_5 finally becomes free, (n_2, n_5) is zeroed and PT is reduced from 9.5 to 9.

It should be pointed out that DSRW is a greedy heuristic that attempts to reduce the parallel time as much as possible. It is always possible that this constraint could prevent the generation of an optimal solution. In Section VI, we examine the performance of DSC on scheduling randomly generated DAG's and compare it with other algorithms.

D. A Running Trace of DSC

We demonstrate the DSC steps by using a DAG example shown in Fig. 1(a). The steps 0, 1, 2, 5, 6, and 7 are shown in Fig. 9(a)–9(f), respectively. The thick paths are the DS's and dashed pseudo edges are the execution order within a cluster. We provide an explanation for each step below. The superscript of a task node in FL or PFL indicates its priority value.

a) Initially, we have the following conditions:

$$\text{UEG} = \{n_1, n_2, n_3, n_4, n_5, n_6, n_7\} \quad PT_0 = 13$$
$$\text{FL} = \{n_1^{0+13}, n_4^{0+9.5}, n_5^{0+7.5}\} \quad \text{PFL} = \{\}.$$

b) n_1 is selected, *tlevel* $(n_1) = 0$, and it cannot be reduced, so CLUST(n_1) remains a unit cluster. Then we have the following condition:

$$\text{UEG} = \{n_2, n_3, n_4, n_5, n_6, n_7\} \quad PT_1 = 13$$
$$\text{FL} = \{n_2^{4+9}, n_3^{1.5+8}, n_4^{0+9.5}, n_5^{0+7.5}\} \quad \text{PFL} = \{\}.$$

c) n_2 is selected, $n_x = n_2$, $n_y =$ NULL, and *tlevel* $(n_2) =$ 4. By zeroing the incoming edge (n_1, n_2) of n_2,

[2]DSC reschedules all predecessors that have been examined and have n_x as the only child. This is the only backtracking currently allowed in DSC without increasing complexity.

tlevel (n_2) reduces to 1. Thus, this zeroing is accepted, and after that step, we have the following condition:

$$\text{UEG} = \{n_3, n_4, n_5, n_6, n_7\} \quad \text{PT}_2 = 10$$
$$\text{FL} = \{n_3^{1.5+8}, n_4^{0+9.5}, n_5^{0+7.5}\} \quad \text{PFL} = \{n_7^{9+1}\}.$$

d) n_3 is selected, $n_x = n_3$ with $\text{PRIO}(n_3) = 1.5 + 8 = 9.5$, and $n_y = n_7$ with $p\text{PRIO}(n_7) = 10$. Notice that by zeroing (n_2, n_7), the *tlevel* (n_7) reduces to 8.5. Thus, we impose the DSRW constraint. Since zeroing (n_1, n_3) affects *tlevel* (n_7), this zeroing is not accepted, because of DSRW. The *tlevel* (n_3) remains the same, and $\text{CLUST}(n_3)$ remains a unit cluster in EG.

e) n_4 and n_5 are selected, respectively, and their clusters again remain unit clusters. After that we have the following condition:

$$\text{UEG} = \{n_6, n_7\} \quad \text{PT}_5 = 10$$
$$\text{FL} = \{n_6^{5+4.5}\} \quad \text{PFL} = \{n_7^{9+1}\}.$$

f) n_6 is selected, and its incoming edges (n_3, n_6) and (n_4, n_6) are zeroed by the minimization procedure. Node n_4 is ordered for execution first, because *tlevel* $(n_4) = 0$, which is smaller than *tlevel* $(n_3) = 1.5$. Then *tlevel* (n_6) is reduced to 2.5, and we have the following condition:

$$\text{UEG} = \{n_7\} \quad \text{PT}_6 = 10$$
$$\text{FL} = \{n_7^{9+1}\} \quad \text{PFL} = \{\}.$$

g) n_7 is selected, and (n_2, n_7) is zeroed so that *tlevel* (n_7) is reduced from 9 to 7. Then we have the following condition:

$$\text{UEG} = \{\} \quad \text{PT}_7 = 8 \qquad \text{FL} = \{\} \quad \text{PFL} = \{\}.$$

Finally, three clusters are generated with PT= 8.

E. DSC Properties

In this subsection, we study several properties of DSC related to the identification of DS, the reduction of the parallel time, and the computational complexity. Theorem 4.1 indicates that DSC (Lines 6 and 9 in Fig. 7) correctly locates DS nodes at each step, even if we use the partial priority *p*PRIO. Theorems 4.2 and 4.3 show how DSC warranties guarantees the reduction of the parallel time. Theorem 4.4 studies the complexity of DSC.

Correctness in Locating DS Nodes: The goal of DSC is to reduce the length of DS's in a sequence of steps. To do that, it must correctly identify unexamined DS nodes. For Lemmas 4.1 and 4.2 and Theorem 4.1, we assume that *a DS goes through UEG*, because it is only then that DSC needs to identify and compress DS. If DS does not go through UEG, but only through EG, then all DS nodes have been examined, and the DS can no longer be compressed, because backtracking is not allowed in general.

Since the DSC algorithm examines the nodes topologically, the free list FL is always nonempty. By definition, all tasks in FL and PFL are in UEG. It is obvious that a DS must go through tasks in either FL or PFL when it goes through UEG. The interesting question is whether a DS also goes through the heads of the priority lists of FL and PFL, because then the algorithm would correctly locate DS nodes for examination. The answer is given in the following lemmas and in Theorem 4.1.

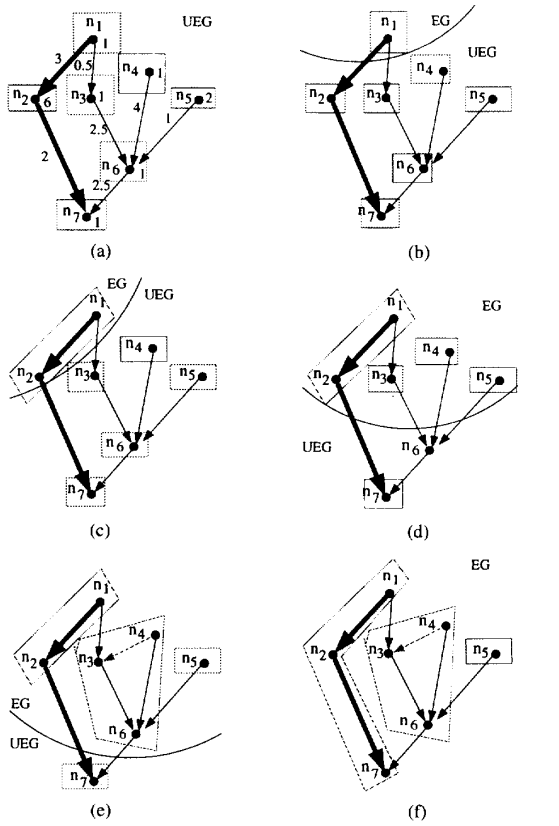

Fig. 9. The result of DSC after each step in scheduling the DAG shown in Fig. 1(a). (a) Step 0. (b) Step 1. (c) Step 2. (d) Step 5. (e) Step 6. (f) Step 7.

Lemma 4.1: Assume that $n_x = \text{head}(FL)$ after step i. If there are DS's going through free nodes in FL, then one DS must go through n_x.

Proof: At the completion of step i, the parallel time is $\text{PT}_i = \text{PRIO}(n_s)$, where n_s is a DS node. Assume that no DS goes through n_x. Then one DS must go through another nonhead free node n_f. This implies that $\text{PRIO}(n_f) = PT_i > \text{PRIO}(n_x)$, which is a contradiction, because n_x has the highest priority in FL.

Lemma 4.2: Assume that $n_y = \text{head}(\text{PFL})$ after step i. If DS's go through only partially free nodes in PFL, then one DS must go through n_y. Moreover, $pPRIO(n_y) = \text{PRIO}(n_y)$.

Proof: First, observe that the starting node of a DS must be an entry node of this DAG. If this node is in UEG, then it must be free, which is impossible, because the assumption says that DS's go through only partially free nodes in UEG. Thus, the starting node must be in EG. As a result, a DS must start from a node in EG and go through an examined node n_j to its unexamined partially free successor n_p. Then, because they are in the same DS, we have the condition that $\text{PRIO}(n_p) = \text{PRIO}(n_j) = p\text{PRIO}(n_p)$. The

proof now becomes similar to the previous lemma. Suppose that no DS goes through n_y. We have the condition that $p\text{PRIO}(n_y) \leq \text{PRIO}(n_y) < \text{PT}_i = \text{PRIO}(n_p) = p\text{PRIO}(n_p)$, which contradicts the assumption that n_y is the head of PFL.

Next we prove that $p\text{PRIO}(n_y) = \text{PRIO}(n_y)$. Suppose $p\text{PRIO}(n_y) < \text{PRIO}(n_y)$. This implies that the DS, to which n_y belongs, does not pass through any examined immediate predecessor of n_y. As a result, DS must go through some other nodes in UEG. Thus, there exists an ancestor node n_a of n_y such that the DS passes through an edge (n_q, n_a), where n_q is the examined predecessor of n_a and n_a is partially free because it is impossible to be free by the assumptions of this lemma. We have the condition that $p\text{PRIO}(n_a) = \text{PRIO}(n_a) = \text{PRIO}(n_y) > p\text{PRIO}(n_y)$, which shows n_y is not the head of PFL. This a contradiction. □

The following theorem shows that the condition $\text{PRIO}(n_x) \geq p\text{PRIO}(n_y)$, used by DSC algorithm in Line 6 of Fig. 7, correctly identifies DS nodes.

Theorem 4.1: Assume that $n_x = \text{head(FL)}$ and $n_y = \text{head(PFL)}$ after step i, and that there is a DS going through UEG. If $\text{PRIO}(n_x) \geq p\text{PRIO}(n_y)$, then a DS goes through n_x. If $\text{PRIO}(n_x) < p\text{PRIO}(n_y)$, then a DS goes through n_y, and there is no DS going through any free node in FL.

Proof: If $\text{PRIO}(n_x) \geq p\text{PRIO}(n_y)$, then we show that a DS goes through n_x. First, assume that DS goes through FL or through both FL and PFL. Then, according to Lemma 4.1, it must go through n_x. Next assume that DS goes through PFL only. Then, from Lemma 4.2, it must go through n_y, implying that $p\text{PRIO}(n_y) = \text{PRIO}(n_y) = \text{PT}_i > \text{PRIO}(n_x)$, which is a contradiction, because $\text{PRIO}(n_x) \geq p\text{PRIO}(n_y)$.

If $\text{PRIO}(n_x) < p\text{PRIO}(n_y)$, suppose that a DS passes a free node. Then, according to Lemma 4.1, $\text{PRIO}(n_x) = \text{PT}_i \geq PRIO(n_y) \geq p\text{PRIO}(n_y)$, which is a contradiction again. Thus, the DS's must go through only partially free nodes, and one of them must go through n_y by Lemma 4.2. □

The Warranty in Reducing Parallel Time: We show that during DSC clustering steps, the parallel time of the clustered graph monotonically decreases. Moreover, after any clustering step i, if there exists an algorithm that could reduce the parallel time of the clustered graph by zeroing some edge, then DSC will guarantee a reduction at some future step.

Lemma 4.3: Assume that $n_x = \text{head(FL)}$ and $n_y = \text{head}(PFL)$ after step i. The following is the parallel time for executing the clustered graph after step i of DSC:

$$\text{PT}_i = \max\{\text{PRIO}(n_x), p\text{PRIO}(n_y), \max_{n_e \in EG}\{\text{PRIO}(n_e)\}\}.$$

Proof: There are three cases in the proof.

1) If DS nodes are only within EG, then, by definition, $\text{PT}_i = \max_{n_e \in \text{EG}}\{\text{PRIO}(n_e)\}$.
2) If a DS goes through a free node, then $\text{PT}_i = \text{PRIO}(n_x)$ by Lemma 4.1.
3) If there is a DS passing through UEG, but this DS passes through only partially free nodes, then $\text{PT}_i = \text{PRIO}(n_y) = p\text{PRIO}(n_y)$, by Lemma 4.2. □

Theorem 4.2: For each step i of DSC, $\text{PT}_{i-1} \geq \text{PT}_i$.

Proof: For this proof, we rename the priority values of $n_x = head(FL)$, and $n_y = head(PFL)$ after step $i-1$, because $\text{PRIO}(n_x, i-1)$ and $p\text{PRIO}(n_y, i-1)$, respectively. We need to prove that $\text{PT}_{i-1} \geq PT_i$, where the following is true:

$$\text{PT}_{i-1} = \max\{\text{PRIO}(n_x, i-1), p\text{PRIO}(n_y, i-1), \max_{n_e \in EG}\{\text{PRIO}(n_e, i-1)\}\}$$

$$\text{PT}_i = \max\{\text{PRIO}(n_x^*, i), p\ \text{PRIO}(n_y^*, i), \max_{n_e \in EG \bigcup \{n_x\}}\{\text{PRIO}(n_e, i)\}\},$$

and n_x^* and n_y^* are the new heads of FL and PFL after step i.

We prove first that $\text{PRIO}(n_x^*, i) \leq \text{PT}_{i-1}$. Since n_x^* is in the free list after step $i-1$, it must be in UEG, and also it must be either the successor of n_x or independent of n_x. At step i, DSC picks up task n_x to examine its incoming edges for zeroing. We consider the effect of such zeroing on the priority value of n_x^*. Since the minimization procedure does not increase *tlevel* (n_x), the length of the paths going through n_x decreases or remains unchanged. Thus, the priority values of the descendants of n_x could decrease, but could not increase. The priority values of other nodes in EG remain the same because the minimization procedure excludes those predecessors of n_x that have children other than n_x. Thus, if n_x^* is the successor of n_x, then $\text{PRIO}(n_x^*, i) \leq \text{PRIO}(n_x^*, i-1)$; otherwise, $\text{PRIO}(n_x^*, i) = \text{PRIO}(n_x^*, i-1)$. Since $\text{PRIO}(n_x^*, i-1) \leq PT_{i-1}$, $\text{PRIO}(n_x^*, i) \leq \text{PT}_{i-1}$. Similarly, we can prove that $\text{PRIO}(n_y^*, i) \leq \text{PT}_{i-1}$.

Next we prove that $\max_{n_e \in \text{EG} \cup \{n_x\}}\{\text{PRIO}(n_e, i)\}\} \leq \text{PT}_{i-1}$. We have $\text{PRIO}(n_x, i) \leq \text{PRIO}(n_x, i-1)$ from the minimization procedure. We need to examine the effect of zeroing only for n_x on the priorities of nodes in EG. The minimization procedure may increase the *tlevel* values of some predecessors of n_x, say, n_p; but it guarantees that the lengths of the paths going through n_p and n_x do not increase. Thus, the new value of $\text{PRIO}(n_p)$ satisfies $\text{PRIO}(n_p, i) \leq \text{PRIO}(n_x, i-1)$. Because $\text{PRIO}(n_x, i-1) \leq \text{PT}_{i-1}$, $\text{PRIO}(n_p, i) \leq \text{PT}_{i-1}$.

The minimization procedure may also increase the *blevel* values of some nodes in the cluster in EG to which n_x is attached. A pseudo edge is added from the last node of that cluster, say, n_e, to n_x if n_e and n_x are independent. If there is an increase in the priority value of n_e, then the reason must be that adding this pseudo edge introduces a new path going through n_x with longer length than the other existing paths for n_e. Since the minimization procedure has considered the effect of attaching n_x after n_e on *tlevel* (n_x), the length of paths will be less than or equal to $\text{PRIO}(n_x, i-1) \leq PT_{i-1}$. Thus, $\text{PRIO}(n_e, i) \leq PT_{i-1}$. Similarly, we can prove that other nodes in $\text{CLUST}(n_e)$ satisfy this inequality. Since the priority values of nodes, say, n_o, that are not the predecessors of n_x, or that are not in $\text{CLUST}(n_e)$, will not be affected by the minimization procedure, we have $\text{PRIO}(n_o, i-1) = \text{PRIO}(n_o, i)$. Thus, $\max_{n_e \in EG \bigcup \{n_x\}}\{\text{PRIO}(n_e, i)\} \leq \text{PT}_{i-1}$. □

Theorem 4.3: After step i of DSC, if the current parallel time is reducible by zeroing one incoming edge of a node

in UEG, then DSC guarantees that the parallel time will be reduced at some step greater or equal to $i+1$.

Proof: The assumption that the parallel time reduces by zeroing *one* edge (n_r, n_s) implies that a DS must go through UEG. This implies that the edge (n_r, n_s) belongs to all DS's that go through UEG; otherwise, the parallel time is not reducible. There are three cases.

1) $n_r \in$ EG and n_s is free. We prove that the node n_s must be the head of FL. If it is not, then another free node in UEG, n_f, must have the same priority and thus belong to a DS. Because all DS's go through (n_r, n_s), n_f must be a successor of n_s. Then n_s and n_f cannot both be free, which is a contradiction.

   Also, since $\text{PT}_i = \textit{tlevel}(n_s) + \textit{blevel}(n_s)$ is reducible by zeroing (n_r, n_s), and since $\textit{blevel}(n_s)$ does not change by such a zeroing, *tlevel* (n_s) is reducible. Thus, during the DSC execution, $n_s = \textit{head}(\text{FL})$ will be picked up at step $i+1$, and since $\textit{tlevel}(n_s)$ is reducible, the minimization procedure will accept the zeroing of (n_r, n_s), and the parallel time will reduce at that step.
2) $n_r \in$ EG and n_s is partially free. We prove that n_s must be the head of PFL. We show it by contradiction. Assume that n_f $(n_f \neq n_s)$ is the head of PFL. Since all DS's go through (n_r, n_s), no free nodes are in DS's, and n_f must be a successor of n_s. Then $p\text{PRIO}(n_f) < \text{PRIO}(n_f)$. But by Lemma 4.2, $p\text{PRIO}(n_f) = p\text{PRIO}(n_f)$, which is a contradiction.

   Assume that $\text{PT}_i = \textit{ptlevel}(n_s) + \textit{blevel}(n_s)$ is reducible by $\delta > 0$ when zeroing (n_r, n_s), and that $\textit{blevel}(n_s)$ does not change by such a zeroing. Then $\textit{ptlevel}(n_s)$ is reducible by at least δ. Thus, during the execution of DSC, the reducibility of $\textit{ptlevel}(n_s)$ will be detected at step $i+1$. Afterward non-DS edges are zeroed until n_s becomes free. However, the reducibility of $\textit{ptlevel}(n_s)$ is not affected by such zeroings, because of DSRW. A non-DS node remains a non-DS node, and no other nodes are moved to $\text{CLUST}(n_r)$. When n_s becomes free, $\textit{ptlevel}(n_s)$ is still reducible by at least δ, and because other SubDS either decrease or remain unchanged, the parallel time is also reducible by at least δ. The rest of the proof becomes the same as in the previous case.
3) $n_r \in$ UEG. Assume that n_r becomes examined at step $j \geq i+1$. At step $i+1$, all DS's must go through (n_r, n_s). If, from $i+1$ to step j, the parallel time has been reduced, then the theorem is true. If the parallel time has not been reduced, then at least one DS has not been compressed, and all DS's still go through (n_r, n_s), because the minimization procedure guarantees that no other DS's will be created. Thus, the parallel time will be reducible at step j by zeroing (n_r, n_s), and the proof becomes the same as in the above cases. □

The following corollary is the direct result of Case 2 above.

Corollary 4.3: Assume that $n_y \in$ PFL, $n_y \in DS$ at step i, and that zeroing of an incoming edge of n_y from its scheduled predecessor would have reduced PT by δ. Then DSC guarantees that when n_y becomes free at step j $(j > i)$, PT can be reduced by at least δ.

The Complexity:

Lemma 4.4: After a node n_s becomes free or partially free, $\textit{blevel}(n_s)$ remains unchanged until n_s is examined. Let n_j be a predecessor of n_s. *tlevel* (n_j) remains unchanged from the step when n_j becomes examined to the step when n_s is examined.

Proof: Referring to Property 3.2 DSC-I, DSC is the same as DSC-I, except that the minimization procedure changes tlevel values of some examined predecessors, say, n_h, of the currently selected free task, say, n_x. But n_h does not have any children other than n_x. Thus, after a node n_s becomes partially free or free, but before it is examined, the *tlevel* value of its examined predecessors will remain unchanged. □

Theorem 4.4: The time complexity of DSC is $O((v+e)\log v)$, and the space complexity is $O(v+e)$.

Proof: The difference in the complexity between DSC-I and DSC results from the minimization procedure within the While loop in Fig. 7. In DSC, we also maintain PFL; but the cost for the v steps is the same $O(v \log v)$ when we use the balanced search trees. Therefore, for Lines 4 and 5 in Fig. 7, v steps cost $O(v \log v)$. For Lines 7 and 8 (or for Lines 9 and 10), the minimization procedure costs $O(|\text{PRED}(n_x)| \log |\text{PRED}(n_x)|)$ at each step. Since $\sum_{n_x \in V} |\text{PRED}(n_x)| = e$ and $|\text{PRED}(n_x)| < v$, v steps cost $O(e \log v)$. When imposing DSRW, the predecessors of a partially free node are checked, and the overall cost is at the most $O(v+e)$.

For Line 13, the *tlevel* values of the successors of n_x are updated. Those successors could be in PFL, and the list needs to be rearranged, because their pPRIO values could be changed. The cost of adjusting each successor in PFL is $O(\log |\text{PFL}|)$, where $|\text{PFL}| < v$. The step cost for Line 13 is $O(|\text{SUCC}(n_x)| \log v)$. Since $\sum_{n_x \in V} |\text{SUCC}(n_x)| = e$, the total cost for maintaining PFL during v steps is $O(e \log v)$.

Also, for Line 13, when one successor becomes free, it needs to be added to FL with cost $O(\log |FL|)$, where $|FL| \leq v$. Since there are a total of v task that could become free during v steps, the total cost in Line 13 spent for FL during v steps is $O(v \log v)$.

Notice that according to Lemma 4.4, after n_x has been moved to EG at Line 13, its *tlevel* value will not affect the priority of tasks in PFL and FL in the rest of the steps. Thus, the updating of FL or PFL occurs only once with respect to each task. Therefore, the time complexity of DSC is $O((e+v)\log v)$. The space needed for DSC is to store the DAG and FL/PFL. The space complexity is $O(v+e)$. □

V. Performance Bounds and Optimality of DSC

In this section, we study the performance characteristics of DS. We give an upper bound for a general DAG and prove the optimality for forks, joins, coarse-grain trees, and a class of fine-grain trees. Since this scheduling problem is NP-complete for a general fine-grain tree and for a DAG that is a concatenation of a fork and a join (series parallel DAG) [3], [5], the analysis shows that DSC not only has a low complexity

but also attains an optimality degree that a general polynomial algorithm could achieve.

A. Performance Bounds for General DAG's

A DAG consists of *fork* (F_x) and/or *join* (J_x) structures such as the ones shown in Figs. 5(a) and 6(a). In [8], we define the grain of DAG as follows.

Let the following conditions exist:

$$g(F_x) = \min_{k=1:m}\{\tau_k\}/\max_{k=1:m}\{c_{x,k}\},$$
$$g(J_x) = \min_{k=1:m}\{\tau_k\}/\max_{k=1:m}\{c_{k,x}\}.$$

Then the *granularity* of G is: $g(G) = \min_{n_x \in V}\{g_x\}$ where $g_x = \min\{g(F_x), g(J_x)\}$.

We call a DAG *coarse grain* if $g(G) \geq 1$. For a coarse-grain DAG, each task receives or sends a small amount of communication compared to the computation of its neighboring tasks. In [8], we prove Theorems 5.1, 5.2, and 5.3.

Theorem 5.1: For a coarse grain DAG, there exists a linear clustering that attains the optimal solution.

Theorem 5.2: Let $\mathrm{PT}_{\mathrm{opt}}$ be the optimum parallel time, and let PT_{lc} be the parallel time of a linear clustering. Then $\mathrm{PT}_{lc} \leq (1 + \frac{1}{g(G)})\mathrm{PT}_{opt}$. For a coarse-grain DAG, $\mathrm{PT}_{lc} \leq 2 \times \mathrm{PT}_{\mathrm{opt}}$.

The following theorem is a performance bound of DSC for a general DAG.

Theorem 5.3: Let $\mathrm{PT}_{\mathrm{dsc}}$ be the parallel time by DSC for a DAG G. Then $\mathrm{PT}_{\mathrm{dsc}} \leq (1 + \frac{1}{g(G)})\mathrm{PT}_{opt}$. For a coarse-grain DAG, $\mathrm{PT}_{\mathrm{dsc}} \leq 2 \times \mathrm{PT}_{\mathrm{opt}}$.

Proof: In the initial step of DSC, all nodes are in the separate clusters, which is a linear clustering. By Theorems 5.2 and 4.2, we have the following conditions:

$$\mathrm{PT}_{\mathrm{dsc}} \leq \cdots \leq \mathrm{PT}_k \leq \cdots \leq \mathrm{PT}_1$$
$$\leq \mathrm{PT}_0 \leq (1 + \frac{1}{g(G)})\mathrm{PT}_{\mathrm{opt}}.$$

For coarse-grain DAG's, the statement is obvious. □

B. Optimality for Join and Fork

Theorem 5.4: DSC derives optimal solutions for fork and join DAG's.

Proof: For a fork, DSC performs the exact same zeroing sequence as DSC-I. The clustering steps are shown in Fig. 5. After n_x is examined, DSC will examine free nodes $n_1, n_2, \cdots, n_m$ in a decreasing order of their priorities. The priority value for each free node is the length of each path $< n_x, n_1 >, \cdots, < n_x, n_m >$. If we assume that $\beta_k + \tau_k \geq \beta_{k+1} + \tau_{k+1}$ for $1 \leq k \leq m-1$, then the nodes are sorted as $n_1, n_2, \cdots, n_m$ in the free list FL.

We first determine the optimal time for the fork, and then show that DSC achieves the optimum. Assume that the optimal parallel time to be $\mathrm{PT}_{\mathrm{opt}}$. If $\mathrm{PRIO}(n_h) = \tau_x + \tau_h + \beta_h > PT_{\mathrm{opt}}$ for some h, then β_i must have been zeroed for $i = 1 : h$; otherwise, we have a contradiction. All other edges $i > h$ need not be zeroed, because zeroing such edge does not decrease PT, but could increase PT. Let the optimal zeroing stopping point be h, and assume that $\beta_{m+1} = \tau_{m+1} = 0$. Then the optimal PT is $\mathrm{PT}_{\mathrm{opt}} = \tau_x + \max(\sum_{j=1}^{h}\tau_j, \beta_{h+1} + \tau_{h+1})$.

DSC zeroes as many as edges possible from left to right up to the point k, as shown in Fig. 5(d), such that $\sum_{j=1}^{k-1}\tau_j \leq \beta_k$ and $\sum_{j=1}^{k}\tau_j > \beta_{k+1}$. We prove that $\mathrm{PT}_{\mathrm{opt}} = \mathrm{PT}_{\mathrm{dsc}}$ by a contradiction. Suppose that $k \neq h$, and $\mathrm{PT}_{\mathrm{opt}} < \mathrm{PT}_{\mathrm{dsc}}$. There are two cases.

1) If $h < k$, then $\sum_{j=1}^{h}\tau_j < \sum_{j=1}^{k}\tau_j \leq \beta_k + \tau_k \leq \beta_{h+1} + \tau_{h+1}$. Thus, $\mathrm{PT}_{\mathrm{opt}} = \tau_x + \beta_{h+1} + \tau_{h+1} \geq \tau_x + \beta_k + \tau_k \geq \tau_x + \max(\sum_{j=1}^{k}\tau_j, \beta_{k+1} + \tau_{k+1}) = \mathrm{PT}_{\mathrm{dsc}}$.
2) If $h > k$, then since $\sum_{j=1}^{h}\tau_j \geq \sum_{j=1}^{k+1}\tau_j > \beta_{k+1} + \tau_{k+1} \geq \beta_{h+1} + \tau_{h+1}$, we have the condtion that $\mathrm{PT}_{\mathrm{opt}} = \tau_x + \sum_{j=1}^{h}\tau_j \geq \tau_x + \max(\sum_{j=1}^{k}\tau_j, \beta_{k+1} + \tau_{k+1}) = \mathrm{PT}_{\mathrm{dsc}}$.

There is a contradiction in both cases.

For a join, the DSC uses the minimization procedure to minimize the *tlevel* value of the root, and the solution and the optimal result for a fork are symmetrical. □

C. Optimality for In-Trees and Out-Trees

An *in-tree* is a directed tree in which the root has outgoing degree 0, and other nodes have the outgoing degree 1. An *out-tree* is a directed tree in which the root has incoming degree 0, and other nodes have the incoming degree 1.

Scheduling in-trees and out-trees is still NP-complete in general as shown by Chretienne [5], and DSC will not give the optimal solution. However, DSC will yield optimal solutions for coarse-grain trees and a class of fine-grain trees.

Coarse-Grain Trees:

Theorem 5.5: DSC gives an optimal solution for a coarse-grain in-tree.

Proof: Since all paths in an in-tree go through the tree root, say, n_x, PT= $tlevel(n_x) + blevel(n_x) = tlevel(n_x) + \tau_x$. We claim that *tlevel* (n_x) is minimized by DSC. We prove it by induction on the depth of the in-tree (d). When $d = 0$, it is trivial. When $d = 1$, it is a join DAG, and *tlevel* (n_x) is minimized. Assume that it is true for $d < k$.

When $d = k$, let the predecessors of root n_x be $n_1, \cdots, n_m$. Since each sub-tree rooted with n_i has depth $< k$ and the disjoint subgraphs cannot be clustered together by DSC, DSC will obtain the minimum *tlevel* time for each n_j where $1 \leq j \leq m$ according to the induction hypothesis.

When n_x becomes free, *tlevel* (n_x) $= \max_{1 \leq j \leq m}\{\mathrm{CT}(n_j) + c_{j,x}\}$, $\mathrm{CT}(n_j) =$ *tlevel*$(n_j) + \tau_j$. Without loss of generality, assume that (n_1, n_x) is in a DS, and that *tlevel* $(n_1) + \tau_1 + c_{1,x}$ has the highest value and *tlevel* $(n_2) + \tau_2 + c_{2,x}$ has the second-highest value. DSC will zero (n_1, n_x), and *tlevel* $(n_x) = \max(\mathrm{CT}(n_1), \max_{2 \leq j \leq m}\{\mathrm{CT}(n_j) + c_{j,x}\})$. (DSC will not zero any more edges, because of the coarse-grain condition. DSC might zero (n_2, n_x) when $g(G) = 1$, but *tlevel* (n_x) does not decrease.)

We need to prove that *tlevel* (n_x) is as small as possible. Because the tree is coarse grain, by Theorem 5.1, linear clustering can be used for deriving the optimal solution. Thus, we can assume that S^* is an optimal schedule that uses

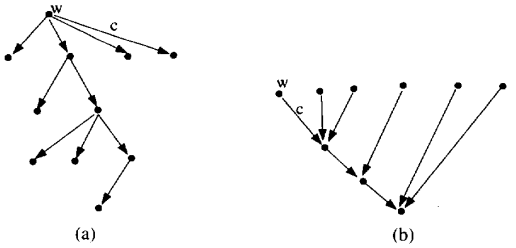

Fig. 10. (a) A single-spawn out-tree. (b) A single-merge in-tree.

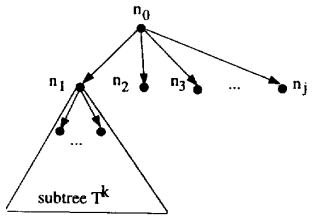

Fig. 11. A single-spawn out-tree named T^{k+1} with height $h = k + 1$.

linear clustering. Let *tlevel* $^*(n_j)$ be the *tlevel* value of n_j in schedule S^*, and let $\text{CT}^*(n_j) = \textit{tlevel}^*(n_j) + \tau_j$, $\text{CT}^*(n_j) \geq \text{CT}(n_j)$ for $1 \leq j \leq m$ from the assumption that $\text{CT}(n_j)$ is minimum. Now we show that *tlevel* $^*(n_x) \geq$ *tlevel*(n_x). There are two cases.

1) If, in S^*, the zeroed incoming edge of n_x is (n_1, n_x), then we have the following conditions:

$$\begin{aligned}\textit{tlevel}^*(n_x) &= \max(\text{CT}^*(n_1), \max_{2\leq j\leq m}\{\text{CT}^*(n_j) + c_{j,x}\}) \\ &\geq \max(\text{CT}(n_1), \max_{2\leq j\leq m}\{\text{CT}(n_j) + c_{j,x}\}) \\ &= \textit{tlevel}(n_x).\end{aligned}$$

2) If, in S^*, the zeroed incoming edge of n_x is not (n_1, n_x), say it is (n_m, n_x). Thus, we have the following condition:

$$\begin{aligned}\textit{tlevel}^*(n_x) &= \max(\text{CT}^*(n_m), \max_{1\leq j\leq m-1}\{\text{CT}^*(n_j) + c_{j,x}\}) \\ &\geq \max(\text{CT}(n_m), \max_{1\leq j\leq m-1}\{\text{CT}(n_j) + c_{j,x}\}).\end{aligned}$$

Because $g(G) \geq 1$, $\text{CT}(n_1) + c_{1,x} \geq \max_{2\leq j\leq m}\{\text{CT}(n_j) + c_{j,x}\}$, we have the following condition: $\textit{tlevel}^*(n_x) \geq \text{CT}(n_1) + c_{1,x} \geq \textit{tlevel}(n_x)$. □

DSC solves an in-tree in time $O(v \log v)$, where v is the number of nodes in this tree. It can be verified that the condition $g(G) \geq 1$ for the in-tree optimality of DSC can be relaxed as $g(J_x) \geq 1$ for all join structures J_x in this tree. Under the same or similar condition, Chretienne [4] and Anger, Hwang, and Chow [2] developed $O(v)$ optimal tree scheduling algorithms. These two algorithms are specific to only this kind of tree.

For an out-tree, the forward clustering of DSC may not find the optimum directly. But for the inverted graph, the backward clustering can obtain the optimum, because an inverted out-tree is an in-tree.

Fine-Grain Trees: Finding optimal solutions for general fine-grain trees is NP-complete. However, DSC is able to obtain optimum for a class of fine-grain trees. A *single-spawn* out-tree is an out-tree such that at most one successor of a nonleaf tree node, say, n_x, can spawn successors. Other successors of n_x are leaf nodes. A *single-merge* in-tree is an inverse of a single-spawn out-tree. Examples of such trees are shown in Fig. 10.

Theorem 5.6: Given a single-spawn out-tree or a single-merge in-tree with an equal computation weight w for each task and an equal communication weight c for each edge, DSC is optimal for this tree.

Proof: We present a proof for an out-tree by induction on the height (h) of this tree. The proof for an in-tree is similar. When $h = 2$, it is a fork, and DSC is optimal. Assume that DSC obtains the optimum when $h = k$.

When $h = k + 1$, we assume, without loss of generality, that the successors of root n_0 are $n_1, n_2, \cdots, n_j$, and that n_1 spawns successors. First, we assume that n_0 has more than one successor; i.e., $j > 1$. Fig. 11 depicts this tree. We call the entire tree T^{k+1}; and the subtree rooted in n_1, T^k. The height of T^k is k, and it has q tasks where $q > 1$. We claim that DSC will examine all nodes in the subtree T^k first before examining other successors of n_0. At step 1, n_0 is examined, and all successors of n_0 become free. Node n_1 has priority $\text{PRIO}(n_1) \geq 3w+2c$, and other successors of n_0 have priority $2w + c$. Then n_1 is examined at step 2, (n_0, n_1) is zeroed, and all successors of n_1 are added to the free list. The priority of n_1's successors $\geq 3w + c$. Thus, they will be examined before $n_2, \cdots, n_j$. Recursively, all of n_1's descendants will be freed and have priority $\geq 3w + c$. Thus, from step 2 to step $q + 1$, all q nodes in T^k are examined one-by-one. After step $q + 1$, DSC looks at $n_2, n_3, \cdots, n_j$.

Since, from step 2 to $q + 1$, DSC clusters T^k only, DSC obtains an optimal clustering solution for this subtree by the induction hypothesis. We call the parallel time for this subtree $\text{PT}_{\text{opt}}(T^k)$. Then the parallel time after step $q + 1$ is $\text{PT}_{\text{dsc}}^{q+1} = \max(w + \text{PT}_{\text{opt}}(T^k), 2w + c)$.

Let PT_{dsc} be the time after the final step of DSC for T^{k+1}. We study the following two cases.

1) One edge in T^k is not zeroed by DSC. Then $\text{PT}_{\text{opt}}(T^k) \geq 2w + c$, implying that $\text{PT}_{\text{dsc}}^{q+1} = w + \text{PT}_{\text{opt}}(T^k)$. Let PT_{dsc} be the time of the final step, because the stepwise parallel time of DSC monotonically decreases, $\text{PT}_{\text{dsc}}^{q+1} \geq \text{PT}_{\text{dsc}}$. Also, since the optimal parallel time for a graph should be no less than that for its subgraph, $\text{PT}_{\text{opt}}(T^{k+1}) \geq w + \text{PT}_{\text{opt}}(T^k)$. Thus, $\text{PT}_{\text{dsc}} \leq \text{PT}_{\text{opt}}(T^{k+1})$, and DSC is optimal for this case.
2) All edges in T^k are zeroed by DSC. Then $\text{PT}_{\text{opt}}(T^k) = qw$, and $\text{PT}_{\text{dsc}}^{q+1} = \max(w + qw, 2w + c)$. If $w + qw \geq 2w + c$, i.e., if $c \leq (q - 1)w$, then $\text{PT}_{\text{dsc}} \leq \text{PT}_{\text{dsc}}^{q+1} = w + \text{PT}_{\text{opt}}(T^k) \leq \text{PT}_{\text{opt}}(T^{k+1})$, because otherwise $c > (q - 1)w$ and $\text{PT}_{\text{dsc}}^{q+1} = 2w + c$. We claim that all edges in T^k and edge (n_0, n_1) should be zeroed by

any optimal clustering for T^{k+1}. If they are not, then $\mathrm{PT}_{\mathrm{opt}}(T^{k+1}) \geq 3w + c > \mathrm{PT}_{\mathrm{dsc}}^{q+1} \geq \mathrm{PT}_{\mathrm{dsc}}$, which is impossible. Since all nodes in T^k and n_0 are in the same cluster, the optimal clustering for the entire out-tree T^{k+1} can be considered as clustering a fork, with "leaf-node" n_1 having a weight qw. Because DSC is optimal for a fork, DSC will get the optimum for T^{k+1}.

Finally, we examine the case in which n_0 has only one successor; i.e., $j = 1$. DSC first zeroes edge (n_0, n_1), and then gets the optimum for T^k. Thus, $\mathrm{PT}_{\mathrm{dsc}} = w + \mathrm{PT}_{\mathrm{opt}}(T^k) = \mathrm{PT}_{\mathrm{opt}}(T^{k+1})$. □

We do not know of another proof of polynomiality of the above class of fine-grain DAG's in the literature. An open question remains: Does there exist a larger class of fine-grain trees that are tractable in polynomial time, say, where the weights are not uniform in the above trees?

VI. A Comparison with Other Algorithms and Experimental Results

There are many clustering algorithms for general DAG's, e.g., [12], [17], and [19]. A comparison of these algorithms is given in [9]. In this section, we compare DSC with the MD algorithm [19] and the ETF [11]. We also provide an experimental comparison for ETF, DSC, and Sarkar's algorithms.

A. The MD Algorithm

Wu and Gajski [19] have proposed two algorithms: the MCP and MD. We refer the reader to [19] for the description of both algorithms, as well as the description of the terms used in this paragraph. The authors use the notion of *as-soon-as-possible* (ASAP) starting time $T_S(n_i)$, the *as-late-as-possible* (ALAP) time $T_L(n_i)$, and the *latest finishing time*, $T_F(n_i)$. The *relative mobility* of a node is defined by $(T_L(n_i) - T_S(n_i))/w(n_i)$, where $w(n_i)$ is the task weight.

The MCP algorithm uses $T_L(n_i)$ as a node priority. It then selects the free node with the smallest node priority, which is equivalent to selecting the node with the largest *blevel*(n_i), and schedules it to a processor that allows its *earliest starting time*.

The MD algorithm uses the relative mobility as a node priority, and, at each step of scheduling, it identifies a task n_p using the smallest relative mobility. It then examines the available processors starting from PE_0, and it schedules n_p to the first processor that satisfies a condition called Fact 1 [19, pp. 336].[3] An intuitive explanation for Fact 1 is that scheduling a task n_p to a processor m should not increase the length of the current critical path (DS).

[3] In a recent personal communication [20], the authors have made the following corrections to the MD algorithm presented in [19]:

1) For Fact 1, when considering processor m, the condition, "for each k," should change to, "There exists k" [19, pp. 336].
2) The T_F and T_S computation [19, pp. 336] should assume that task n_p is scheduled on processor m.
3) When n_p is scheduled to processor m, n_p is inserted before the first task in the task sequence of processor m that satisfies the inequality listed in Fact 1.

TABLE II
A Comparison of MCP, ETF, MD, and DSC

	MCP [19]	ETF [11]	MD [19]	DSC
Task priority	*blevel*	earliest task first (*tlevel*)	relative mobility	*tlevel* + *blevel*
DS task first	no	no	yes	yes
Processor selection	processor for earliest starting	processor for earliest starting	first processor satisfying Fact 1	minimization procedure
Complexity	$O(v^2 \log v)$	$O(pv^2)$	$O(pv(v+e))$	$O((v+e)\log v)$
Join/Fork	no	no	optimal	optimal
Coarse grain in-tree	optimal	optimal	no	optimal

The complexity of the original algorithm is $O(v^3)$, as shown in [19, pp. 337]. The corrected version of MD has a better performance, but slightly higher complexity. This is because of the recomputation of T_S and T_F for each scanned processor. For each scheduling step, the complexity of MD is $O(p(v+e))$, where p is the number of scanned processors, and $O(v+e)$ for the mobility information, and checking Fact 1 for each processor, and because there are v steps, the total complexity for the revised MD is $O(pv(v+e))$.

The idea of identifying the important tasks in DSC is the same as in MD, i.e., the smallest relative mobility identifies DS nodes that have the maximum *tlevel* + *blevel*. However, the way to identify DS nodes is different. The DSC uses a priority function with an $O(\log v)$ computing scheme, whereas MD uses the relative mobility function with a computing cost of $O(v+e)$. Another difference is that when DSC picks a DS task n_p to schedule, it uses the minimization procedure to reduce the *tlevel* of this task and thus decrease the length of DS going through this task. On the other hand, the MD scans the processors from left to right to find the first processor satisfying Fact 1. Even though Fact 1 guarantees the nonincrease of the current critical path, it does not necessarily make the path length shorter.

For a fork or join, MD picks up the DS node at each step, and we can show that it produces an optimal solution. For a coarse-grain tree, as we saw in our optimality proof, the *tlevel* must be reduced at each step. Since the MD schedules a task to a processor that does not necessarily decrease the *tlevel* at each step, the MD may not produce the optimum in general. A summary of this comparison is given in Table II.

B. The ETF Algorithm

ETF [11] is a scheduling algorithm for a bounded number of processors with arbitrary network topologies. At each scheduling step, ETF finds a free task whose starting time is the smallest, and then assigns this task to a processor in which the task execution can be started as early as possible. If there is a tie, then the task with the highest *blevel* is scheduled; in [11], this heuristic is called the ETF/CP. ETF is designed for scheduling on a bounded number of processors. Thus, to compare the performance of DSC and ETF, we first apply DSC to determine the number of processors (clusters), which we then use as an input to the ETF algorithm.

We discuss the differences and similarities of DSC and ETF as follows. For a node priority, DSC uses *tlevel* + *blevel* and then selects the *largest* node priority. On the other hand, the ETF uses the *earliest task first*, which is similar to using *tlevel* as node priority and then selecting the *smallest* node

priority for scheduling. For the scheduling step DSC and ETF, use the same idea, i.e., try to reduce *tlevel* by scheduling to a processor that can start a task *as early as possible*. However, the technique for choosing a processor is different. ETF places a task to the processor that allows the earliest starting time without rescheduling its predecessors, whereas DSC uses the minimization procedure that could reschedule some of the predecessors. It should be mentioned that the MCP [19] algorithm also schedules a task to a processor that allows its earliest starting time as in ETF. However, the node priority for MCP is *blevel*, as opposed to *tlevel* used by ETF.

The complexity of ETF is higher than that of DSC. Since at each step ETF examines all free tasks on all possible processors to find the minimum starting time, the complexity of ETF is $O(pw)$, where p is the number of processors used and w is the maximum size of the free task list. For v tasks, the total complexity is $O(pwv)$. In our case, $p = O(v)$ and $w = O(v)$; thus, the worst complexity is $O(v^3)$. We have used the balanced searching tree structure for the ETF algorithm in finding the values of a clock variable, NM, [11, pp. 249–250]. However, the complexity of ETF for finding the earliest task at each step cannot be reduced, because the earliest starting time of a task depends on the location of processors to be assigned. In practice, the average complexity of ETF could be lower than $O(v^3)$. For the Choleski decomposition DAG described in Section VI-C, $p = O(\sqrt{v})$ and $w = O(\sqrt{v})$; thus, the actual complexity is $O(v^2)$. In Section VI-C, we compare the central processing unit (CPU) time spent for DSC and ETF on a SUN–4 workstation.

For a join DAG, ETF does not use a minimization procedure such as DSC, and it may not be optimal. For a fork, ETF may pick up a task with the earliest starting time; but this task may not be in a DS, and thus ETF may not give the optimum. For a coarse-grain in-tree, ETF places a task to the processor of its successor, which allows the earliest starting time for this task. We can use a similar approach to that in DSC to prove the optimality of ETF for coarse-grain in-trees.

Table II offers a summary of the comparison. Notice the similarities and differences between MCP and ETF and between MD and DSC. For a detailed comparison of MCP and DSC, see [9].

C. Random DAG's

Because of the NP-completeness of this scheduling problem, heuristic ideas used in DSC cannot always lead to an optimal solution. Thus, it is necessary to compare the average performance of different algorithms by using randomly generated graphs. Since both the MD and DSC are using the DS to identify the important tasks, we expect a similar performance from both methods. On the other hand, ETF, DSC, and Sarkar's are based on different principles, and it is of interest to conduct an experimental comparison of these three methods.

We have generated 180 random DAG's as follows. We first randomly generate the number of layers in each DAG. We then randomly place a number of independent tasks in each layer. Next we randomly link the edges between tasks at different layers. Finally, we assign random values to task and edge weights. The following statistic information is important for analyzing the performance of scheduling algorithms:

W: the range of independent tasks in each layer, which approximates the average degree of parallelism;

L: the number of layers; and

R/C: the average ratio of task weights over edge weights. It approximates the graph granularity.

The 180 graphs are classified into three groups of 60 graphs, each based on their R/C values.

M1: The R/C range is 0.8–1.2. The average weights of computation and communication are close.

M2: The R/C range is 3–10. The graphs are coarse grain.

M3: The R/C range is 0.1–0.3. The graphs are fine grain.

Each group is further classified into six subgroups with 10 graphs each, based on the values of W and L. The results of scheduling groups M1, M2, and M3 are summarized in Tables III, IV, and V. The fifth and sixth columns of the tables show the parallel time improvement ratio of DSC over ETF and Sarkar's algorithm. The improvement ratio of DSC over algorithm A is defined as follows:

$$\mathrm{DSC}/A = 1 - \frac{\mathrm{PT}_{\mathrm{dsc}}}{\mathrm{PT}_A}.$$

For group M1, DSC/ETF shows an improvement by 3%, and DSC/Sarkar's shows an improvement by 20%. For the coarse-grain group M2, the performance differences are insignificant between ETF and DSC and small between Sarkar's and DSC. For the fine-grain group M3, the performance is similar to that of M1, except in the first subgroup, where Sarkar's performs the best. This is because, for this group, the degree of parallelism and the number of layers are small, and the granularity is relatively fine. This implies that communication dominates the computation, and because Sarkar's algorithm reduces the communication volume by zeroing the largest communication edge at each step, it can get the largest reduction sooner. This is not the case for the other subgroups, because then the size of graphs is larger, and DSC is given more opportunities (steps) to zero edges.

A summary of the experiments for the 180 random DAG's is given in Table VI. We list the percentage of cases in which the performance of DSC is better, the same, and worse than that of the other two algorithms. This experiment shows that the average performance of DSC is better than that of Sarkar's, and is slightly better than that of ETF.

To see the differences in complexity between DSC and ETF, and to demonstrate the practicality of DSC, we consider an important DAG in numerical computing, the Choleski

TABLE III
DSC VS. ETF AND SARKAR'S FOR GROUP M1
(R/C range is between 0.8 and 1.2)

Layer range	Width Avg/Max	#tasks range	#edge range	DSC/ETF avg	DSC/Sarkar avg
9-11	4/11	44-94	57-206	4.58%	15.73%
9-11	9/20	64-107	118-255	4.49%	17.49%
18-21	5/11	84-121	131-276	4.23%	18.59%
18-20	9/22	158-210	334-552	3.27%	23.28%
18-20	19/41	313-432	691-1249	1.95%	23.40%
36-40	11/22	397-618	900-2374	1.30%	25.93%
Average				3.30%	20.74%

TABLE IV
DSC VS. ETF AND SARKAR'S FOR GROUP M2
(R/C range is between 3 and 10, coarse-grain DAG's)

Layer range	Width Avg/Max	#tasks range	#edge range	DSC/ETF avg	DSC/Sarkar avg
9-11	4/11	26-76	40-149	0.26%	3.60%
9-11	8/21	69-101	115-228	0.07%	5.04%
18-21	5/10	93-115	201-331	0.02%	5.56%
19-21	10/22	172-247	383-833	0.04%	6.16%
18-20	20/41	255-441	571-1495	0.02%	6.30%
35-41	11/23	378-504	676-1452	-0.03%	6.67%
Average				0.06%	5.56%

TABLE V
DSC VS. ETF AND SARKAR'S FOR GROUP M2
(R/C range is between 0.1 and 0.3, fine-grain DAG's)

Layer range	Width Avg/Max	#tasks range	#edge range	DSC/ETF avg	DSC/Sarkar avg
9-10	4/11	38-68	34-187	-2.90%	-5.10%
9-11	9/21	54-118	83-257	2.79%	10.58%
18-20	5/12	84-153	154-469	-0.13%	18.43%
19-21	10/22	181-277	441-924	3.00%	25.96%
18-20	20/41	346-546	843-1992	3.78%	33.04%
35-41	11/23	391-474	632-1459	7.62%	33.41%
Average				2.36%	19.39%

TABLE VI
A SUMMARY OF THE PERFORMANCE OF THREE ALGORITHMS FOR THE 180 DAG'S

	DSC vs. ETF	DSC vs. Sarkar
Avg DSC/A	1.91%	15.23%
#cases, better	56.67%	93.89%
#cases, same	23.33%	0.00%
#cases, worse	20.00%	6.11%

TABLE VII
DSC VS. ETF FOR THE CD DAG
(CPU is time spent for scheduling on Sun4 workstation)

n	v	e	DSC/ETF	CPU_{dsc}	CPU_{etf}
10	55	90	5.00%	0.06 sec	0.05 sec
20	210	380	4.15%	0.18 sec	0.63 sec
40	820	1560	2.47%	0.70 sec	10.4 sec
80	3240	6320	1.33%	3.01 sec	171.0 sec
160	12880	25440	0.69%	13.1 sec	2879 sec
320	51360	102080	0.35%	56.8 sec	55794sec (15 hrs)

decomposition (CD) DAG [7]. For a matrix of size n, the degree of parallelism is n, the number of tasks v is about $n^2/2$, and the number of edges e is about n^2. The average R/C is 2. The performance of the two algorithms is given in Table VII. We show the PT improvement, as well as the total CPU time spent in scheduling this DAG on a Sun-4 workstation.

To explain why the values of the CPU in Table VII makes sense for different values of n, we examine the complexity of the algorithms for this DAG. The complexity of DSC is $O((v+e)\log v)$, which is $O(n^2 \log n)$ for this case. When n increases by 2, CPU_{dsc} increases by about four times. For ETF, the complexity is $O(\text{pvw})$. For this case, $p = w = n$ and $v = n^2/2$; thus, the complexity is $O(n^4)$. When n increases by 2, CPU_{etf} increases by about 16.

VII. CONCLUSION

We have presented a low-complexity scheduling algorithm with performance comparable to, or even better, on average, than much-higher-complexity heuristics. The low complexity makes DSC very attractive in practice. DSC can be used in the first step of Sarkar's [17] two-step approach to scheduling on a bounded number of processors. We have already incorporated DSC into our programming environment PYRROS [23] that has produced very good results on real architectures such as nCUBE-II and INTEL/i860. DSC is also useful for partitioning and clustering of parallel programs [17], [21]. A particular area in which DSC could be useful is scheduling irregular task graphs. Pozo [16] has used DSC to investigate the performance of sparse matrix methods for distributed memory architectures. Wolski and Feo [18] have extended the applicability of DSC to program partitioning for NUMA architectures.

ACKNOWLEDGMENT

We are very grateful to V. Sarkar for providing us with the programs of his system in [17] and for his comments and suggestions, and to M.-Y. Wu for his help in clarifying the MD algorithm. We also thank referees for their useful suggestions in improving the presentation of this paper. We thank W. Wang for programming the random graph generator, and A. Darte, P. Sadayappan, and R. Wolski for their comments on this work. The analysis for a fine-grain tree was inspired by a conversation with T. Varvarigou.

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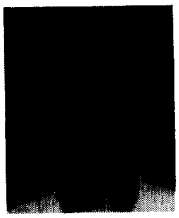

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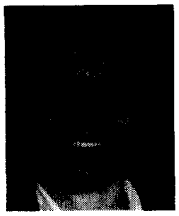

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