CS 4824/ECE 4424: Linear Regression

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Regression

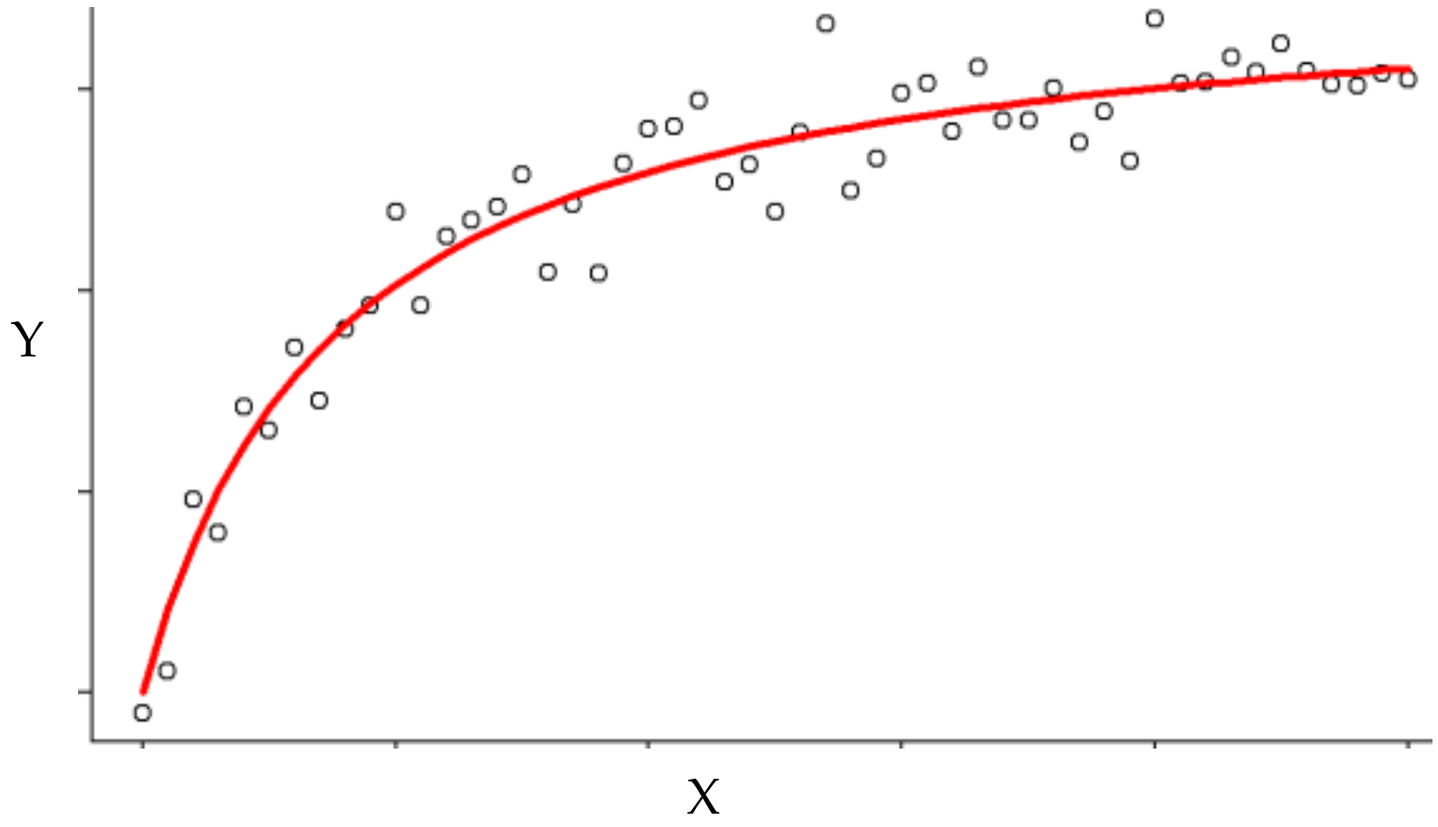
- So far, we've been interested in learning $P(Y|X)$ where Y has discrete values (called 'classification')
- What if Y is continuous? (called 'regression')
  - predict snow/rainfall from current and past weather features
  - predict stock price from current and past market conditions
  - predict weight from gender, height, age, ...

Regression: problem setting

- Wish to learn f: $X \rightarrow Y$ where Y is real-valued, given training data $\{<X^1, Y^1> \ldots <X^n, Y^n>\}$

- Approach:
  - Choose some parameterized form for $P(Y \mid X, \theta)$ where θ is the vector of parameters
  - Estimate θ using MLE or MAP estimation

Choose parameterized form for $P(Y|X,\theta)$



- Assume Y is some deterministic f(X), plus random noise
  - $y = f(x) + \varepsilon$ where $\varepsilon \sim \mathcal{N}(0,\sigma)$
- Therefore, Y is a random variable that follows the distribution
  - $p(y|x) = \mathcal{N}(f(x),\sigma)$
- And the expected value of y for any given x is f(x)

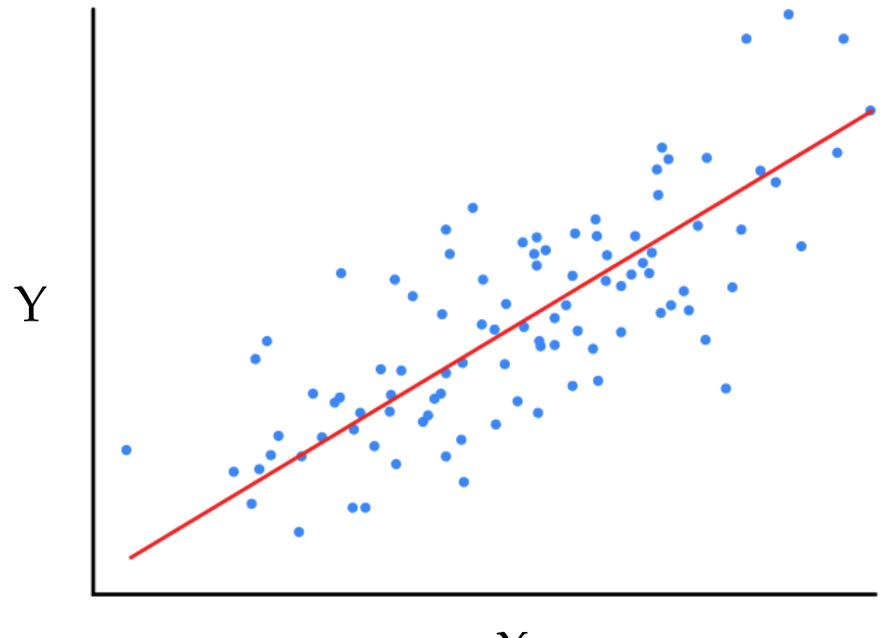
Consider linear regression

- $p(y \mid x) = \mathcal{N}(f(x), \sigma)$

- Assume f(x) is a linear function of
  i.e.,

- $p(y \mid x) =$

- $\mathbb{E}(y \mid x) =$



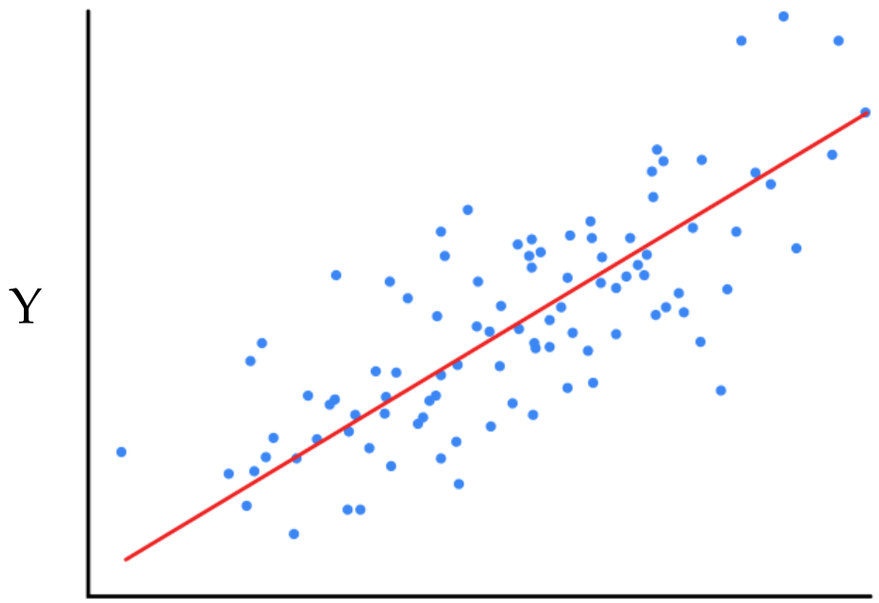
Consider linear regression

- $p(y \mid x) = \mathcal{N}(f(x), \sigma)$

- Assume f(x) is a linear function of x

  i.e., $y = w_0 + w_1 x$

  - $p(y \mid x) = \mathcal{N}(w_0 + w_1 x, \sigma)$
  - $\mathbb{E}(y \mid x) = w_0 + w_1 x$

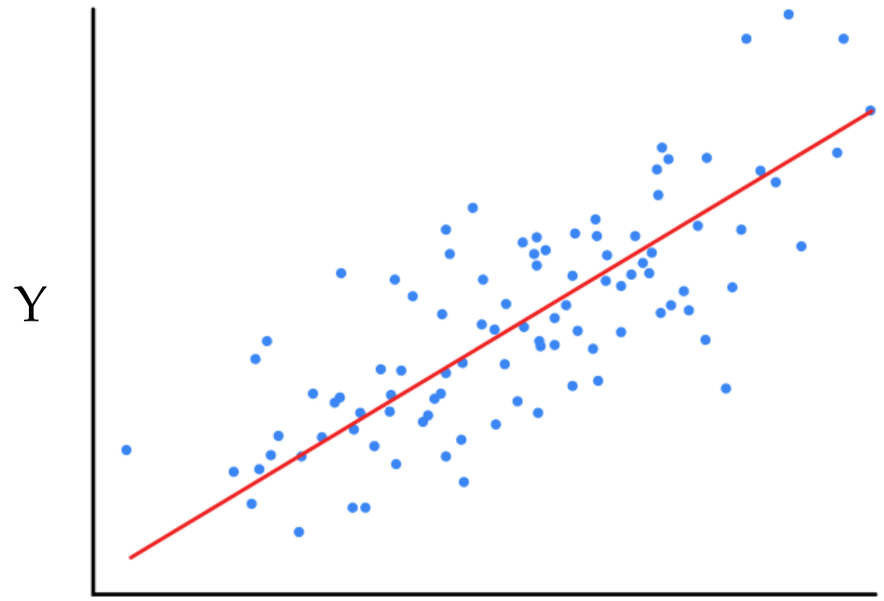


- Note: to make our parameters explicit, let's write
  - $W = < w_0, w_1 >$
  - $p(y \mid x, W) = \mathcal{N}(w_0 + w_1 x, \sigma)$

Training linear regression

- $p(y \mid x) = \mathcal{N}(f(x), \sigma)$

- How can we learn W from data?



- Learn W using Maximum Conditional Likelihood Estimation!
  - $W_{MCLE} = \arg\max_W \prod_l P(Y^l \mid X^l, W)$
  - $W_{MCLE} = \arg\max_W \sum_l \ln p(y^l \mid x^l, W)$
  - Where $P(y \mid x, W) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{y^l - f(x, W)}{\sigma}\right)^2\right)}$

MCLE derivation

- $W_{MCLE} = \arg\max_W \sum_l \ln p(y^l \mid x^l, W)$
- $P(y \mid x, W) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{y^l - f(x,W)}{\sigma}\right)^2\right)}$ or $P(y \mid x, W) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{y^l - (w_0 + w_1 x^l)}{\sigma}\right)^2\right)}$

Training linear regression

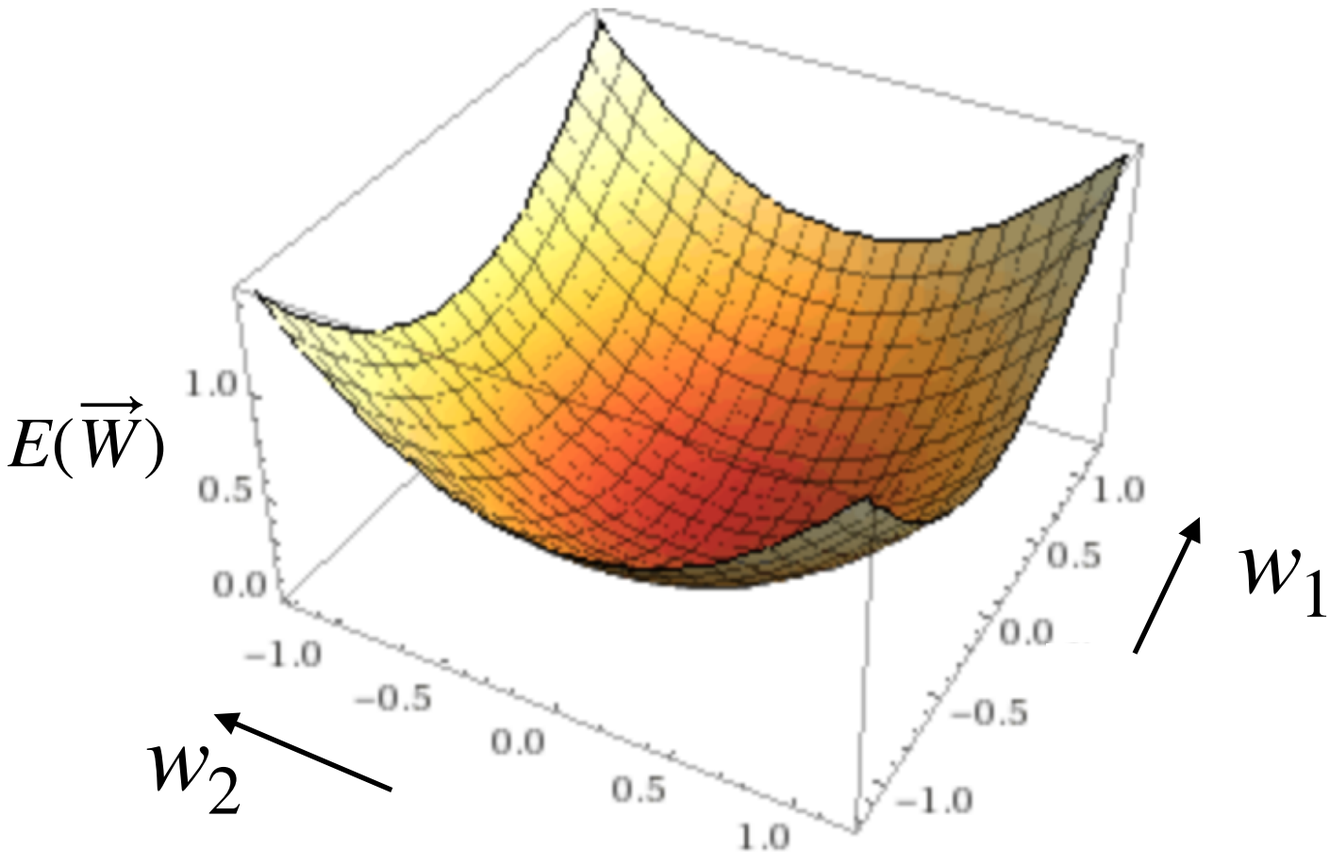
- Learn W using Maximum Conditional Likelihood Estimation:

$$W_{MCLE} = \arg\min_{W} \sum_{l} (y - f(x, W))^2$$

- This corresponds to minimizing sum of squared errors (often used in "curve fitting")

- How to perform the minimization in order to choose optimal $W = < w_0, w_1 >$?

Gradient descent



- **Gradient** $\nabla E(\vec{w}) = [\frac{\partial E}{\partial w_0}, \frac{\partial E}{\partial w_1}, \ldots, \frac{\partial E}{\partial w_n}]$
  - Training Rule: $\vec{w}^{(i+1)} \leftarrow \vec{w}^{i} - \eta \nabla E(\vec{w})$
  - i.e. $\Delta w_i = -\eta \frac{\partial E}{\partial w_i}$

Minimizing squared error: gradient descent

- $$\frac{\partial E}{\partial w_1} = \sum_l 2(y^l - (w_o + w_1 x^l))(-x^l)$$
- $$= -2 \sum_l (y^l - (w_o + w_1 x^l))(x^l)$$
- $$\frac{\partial E}{\partial w_0} = -2 \sum_l (y^l - (w_o + w_1 x^l))$$

Minimizing squared error: gradient descent

- $\frac{\partial E}{\partial w_1} = -2\sum_l (y^l - (w_o + w_1 x^l))(x^l)$ $\frac{\partial E}{\partial w_0} = -2\sum_l (y^l - (w_o + w_1 x^l))$
- Update rule:

Linear regression more generally

- $p(y \mid x) = \mathcal{N}(f(x), \sigma I) \qquad \overrightarrow{X} = < x_1, x_2, \ldots, x_n >$
- $f(x) = w_0 + \sum_{i=1}^{n} w_i x_i$
- $p(y \mid x) = \mathcal{N}(w_0 + \sum_{i=1}^{n} w_i x_i, \sigma I)$
- $\mathbb{E}(y \mid x) = w_0 + \sum_{i=1}^{n} w_i x_i$
- $p(y \mid x, W) = \mathcal{N}(w_0 + \sum_{i=1}^{n} w_i x_i, \sigma I) \qquad \overrightarrow{W} = < w_0, w_1, \ldots, w_n >$

Minimizing squared error more generally: gradient descent

- **Gradient descent algorithm**: iterate until change < ε
  - $\forall\, i$ repeat $\quad w_i \leftarrow w_i + 2\eta \sum_l X_i^l (Y^l - (w_0 + \sum_{j=1}^{n} w_j x_j))$
  - assume $\quad X_0 = 1$ for w_0

How about MAP estimation?

- $W_{MAP} = \arg\max_{W}(-c\sum_{l} w_i^2) + \sum_{l} \ln P(Y^l \mid X^l, W)$

  - Called a "**regularization**" term
  - Helps reduce overfitting, especially for sparse data situations
  - Keeps weights near zero with prior $W \sim \mathcal{N}(0, \sigma I)$, or whatever the prior suggests

Demo Time 🙂

https://lukaszkujawa.github.io/gradient-descent.html

Summary of regression

- Assuming $p(y \mid x, W) = \mathcal{N}(w_0 + w_1 x, \sigma)$
  - MLE corresponds to minimizing sum of squared errors
  - MAP estimate minimizes SSE plus sum of squared weights
  - Again, learning is an optimization problem once we choose our objective function
    - maximize data likelihood
    - maximize posterior prob of W
- Again, we can use gradient descent as a general learning algorithm
- Be careful about outliers while performing regression
- **NOTE**: Almost nothing here required that f(x) be linear in x