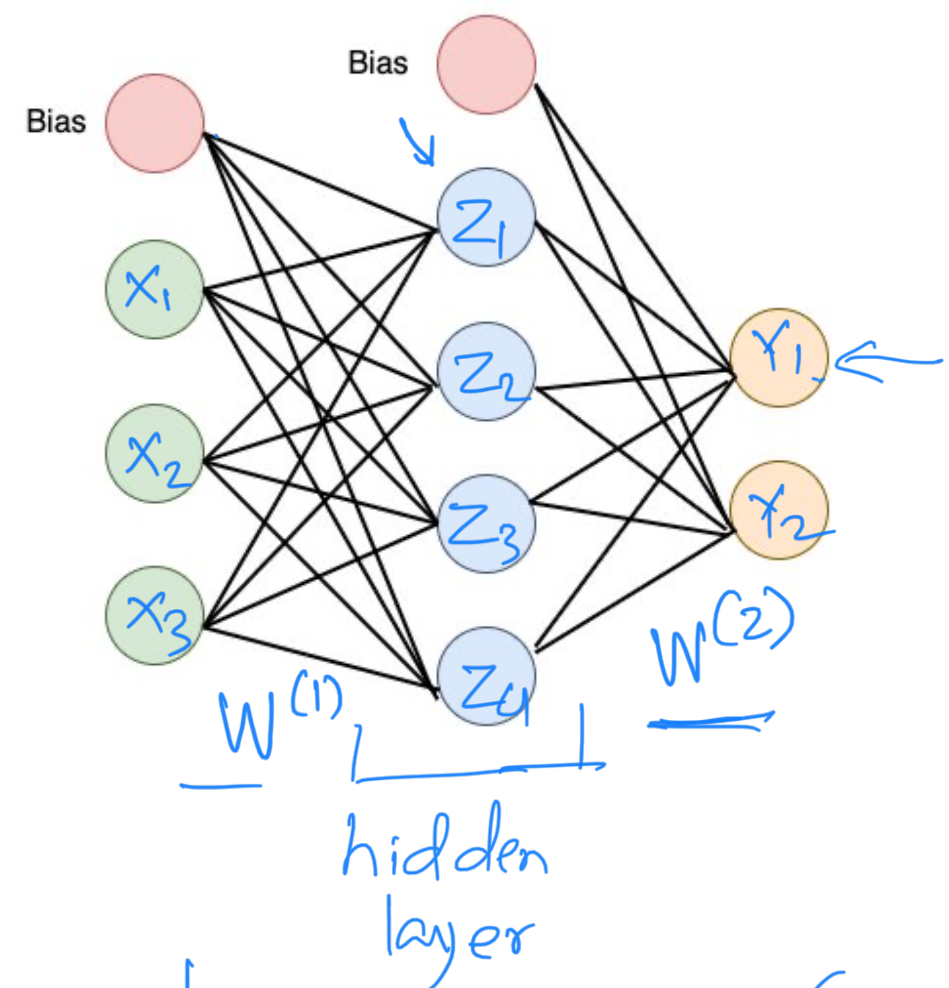
CS 4824/ECE 4424: Neural Networks

Acknowledgement:
Many of these slides are derived from Tom Mitchell, Pascal Poupart, Pieter Abbeel, Eric Eaton, Carlos Guestrin, William Cohen, and Andrew Moore.

Two-layer Feed-forward Network

- Architecture



- Hidden nodes: $z_j = h_1\,(\mathbf{w_j}^{(1)T}\,\boldsymbol{x})$ $= h_1\left(\sum_i W_{ji}^{(1)} x_i\right)$
- Output nodes: $y_k = h_2\,(\mathbf{w_k}^{(2)T}\,\boldsymbol{z})$ $= h_2\left(\sum_j w_{kj}^{(2)} z_j\right)$
- Overall: $\boxed{y_k} = h_2(\sum_j w_{kj}^{(2)}\, h_1(\sum_i w_{ji}^{(1)}\, x_i))$

Common Activation Functions h

- Identity $h(a) = a$
- Threshold $h(a) = \begin{cases} 1 & if\ a \geq 0 \\ 0 & if\ a < 0 \end{cases}$
- Sigmoid $h(a) = \sigma(a) = \dfrac{1}{1 + e^{-a}}$
- Gaussian $h(a) = e^{-\frac{1}{2}\left(\frac{a-\mu}{\sigma}\right)^2}$
- Tanh $h(a) = tanh(a) = \dfrac{e^{a} - e^{-a}}{e^{a} + e^{-a}}$

Two-layer Feed-forward Network

- Regression

$$y_K = \left(\sum_j w_{Kj}^{(2)} \underbrace{\sigma\left(\sum_i w_{ji}^{(1)} x_i\right)}_{\text{non-linear functions}}\right)$$

linear combination

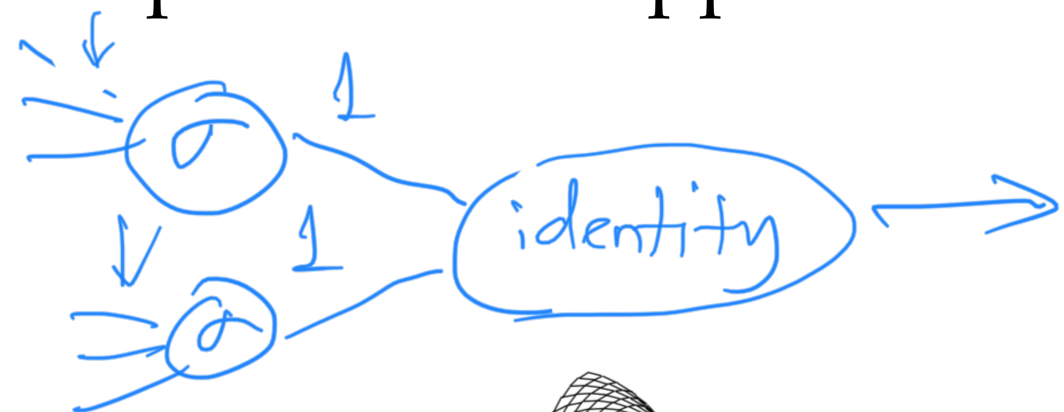
- Classification

$$P(y_K|x) = \sigma\left(\sum_j w_{Kj}^{(2)} \sigma\left(\sum_i w_{ji}^{(1)} x_i\right)\right)$$

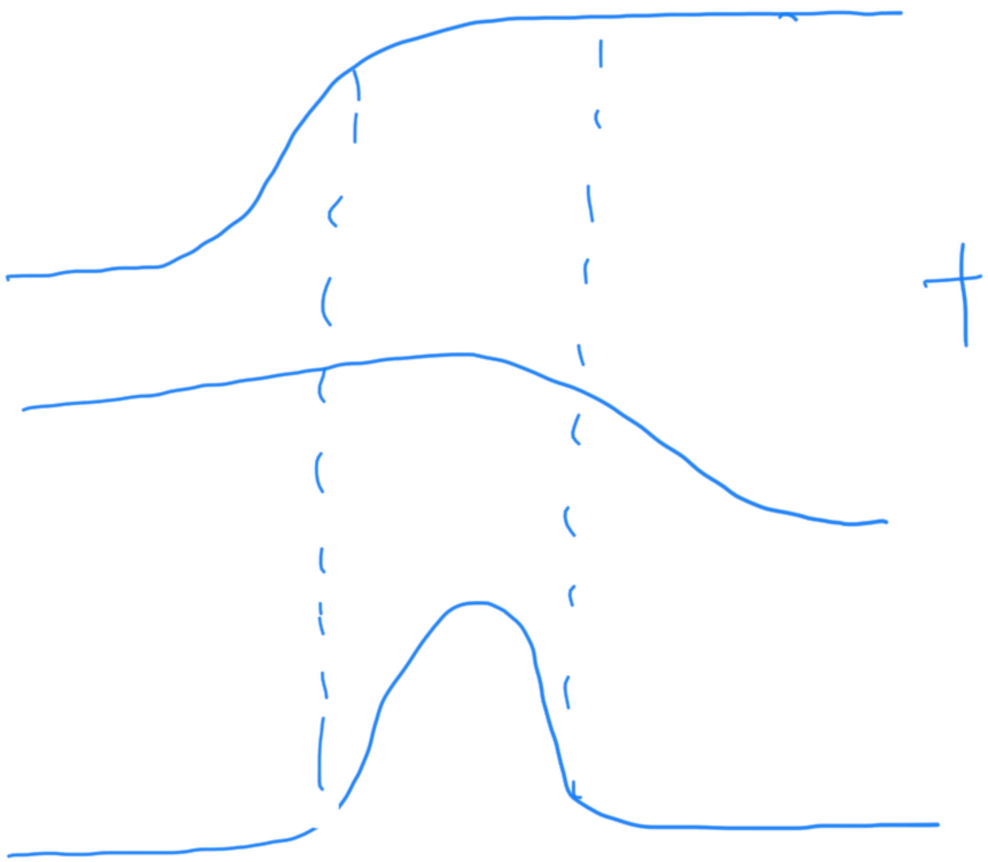
return a prob.

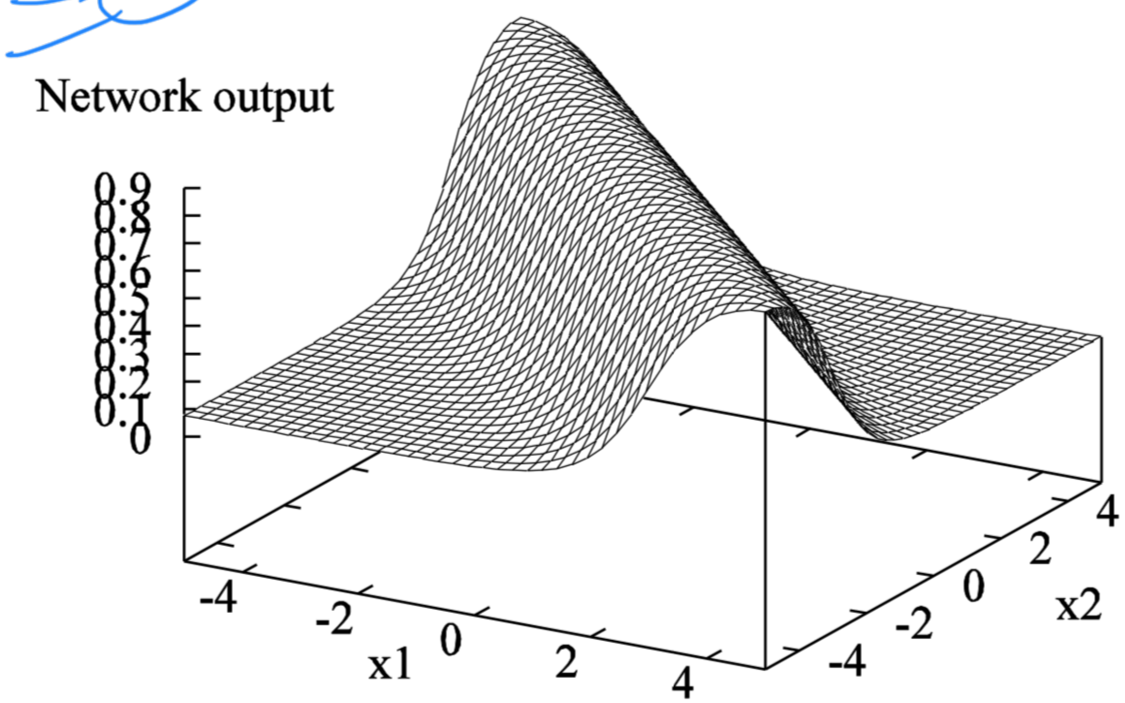
Combining Activation Functions

- Adding two sigmoid nodes with parallel but opposite "cliffs" produces a **ridge**



- Schematic

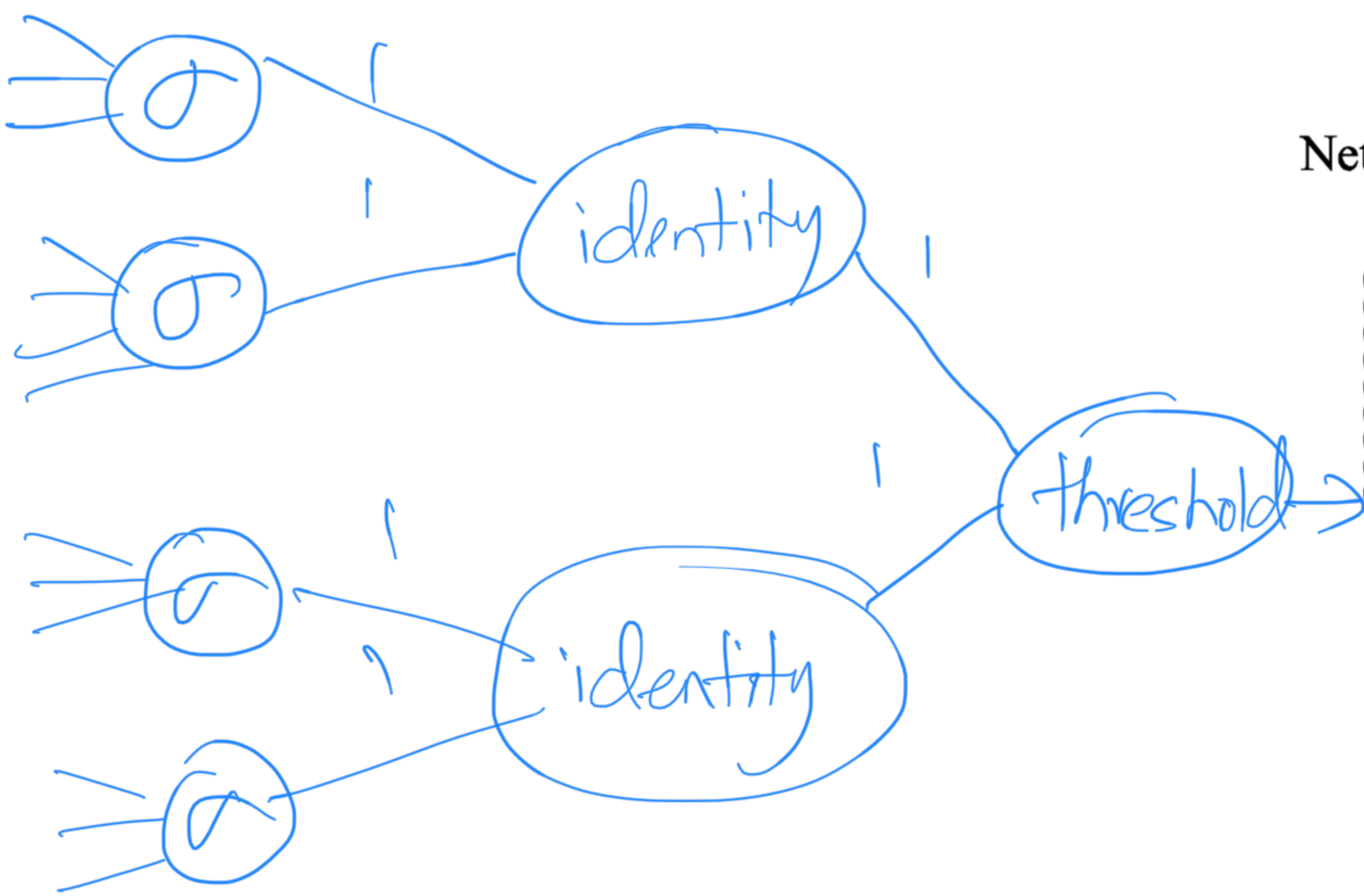


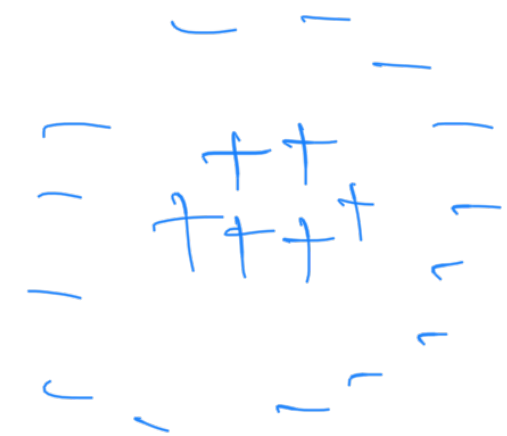


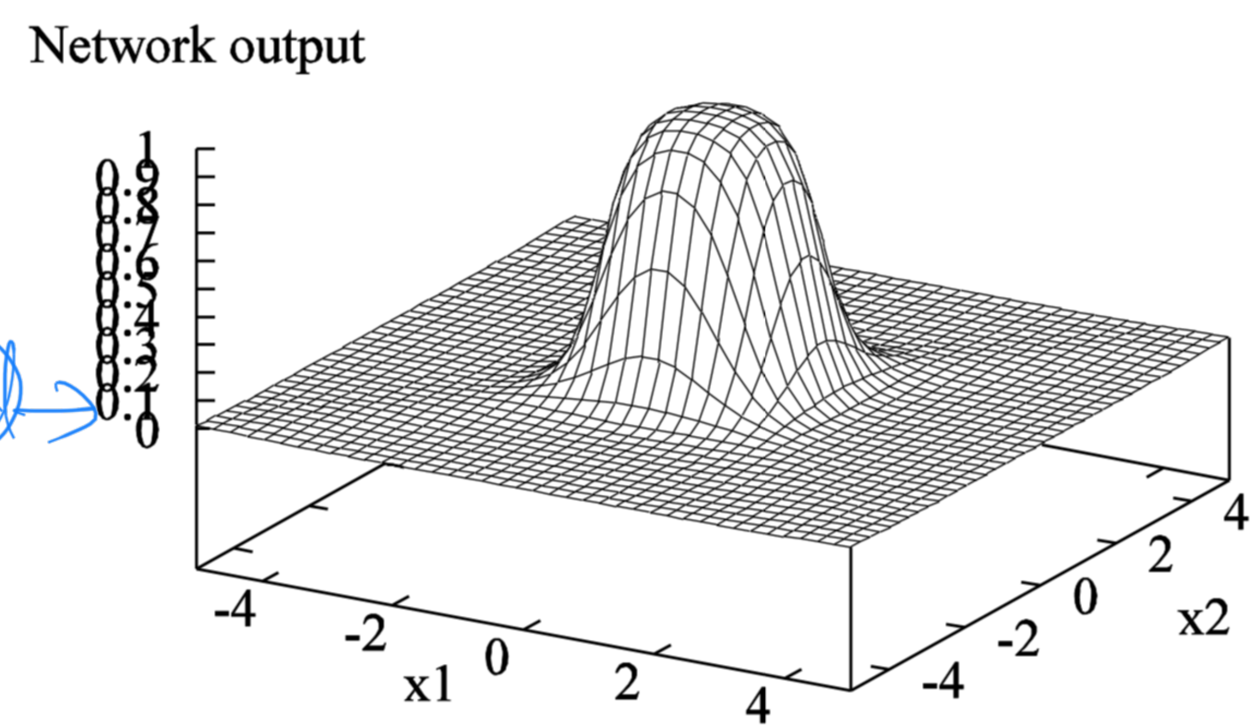
Combining Activation Functions

- Adding two intersecting ridges (and thresholding) produces a **bump**

- Schematic

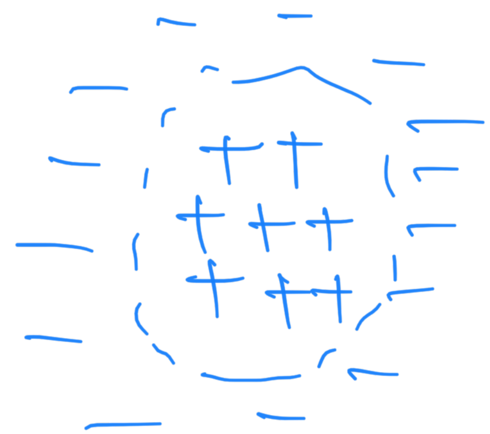




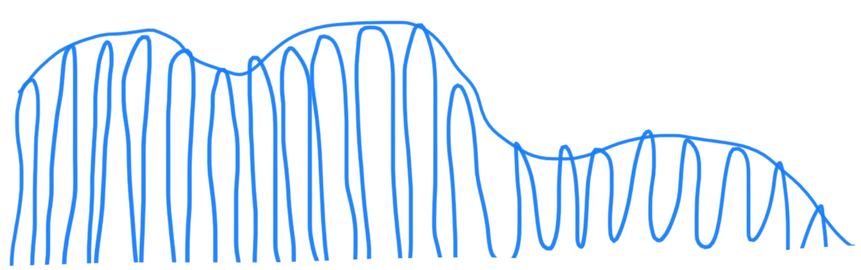


Combining Activation Functions

- A bump can classify linearly non-separable data points



- By tiling bumps of various heights together, we can approximate any function



Combining Activation Functions

- Combining activation functions in a neural network enables us to approximate any function, hence millions of applications

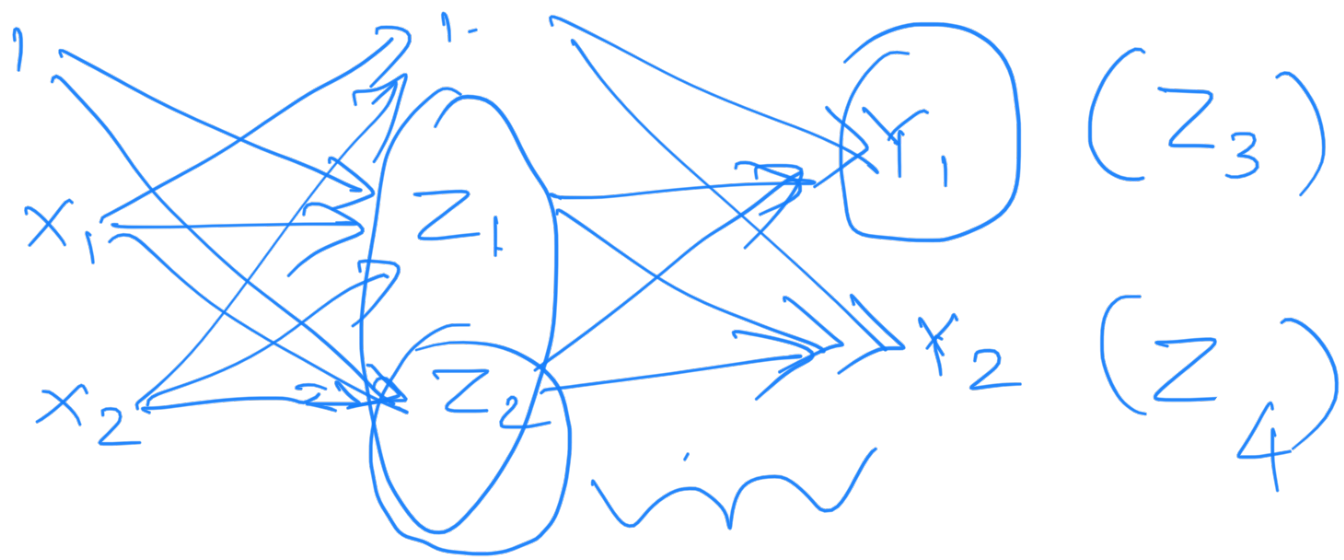
  - Machine translation
  - Computer vision
  - Speech recognition
  - Word embedding
  - …

Optimizing the Weights

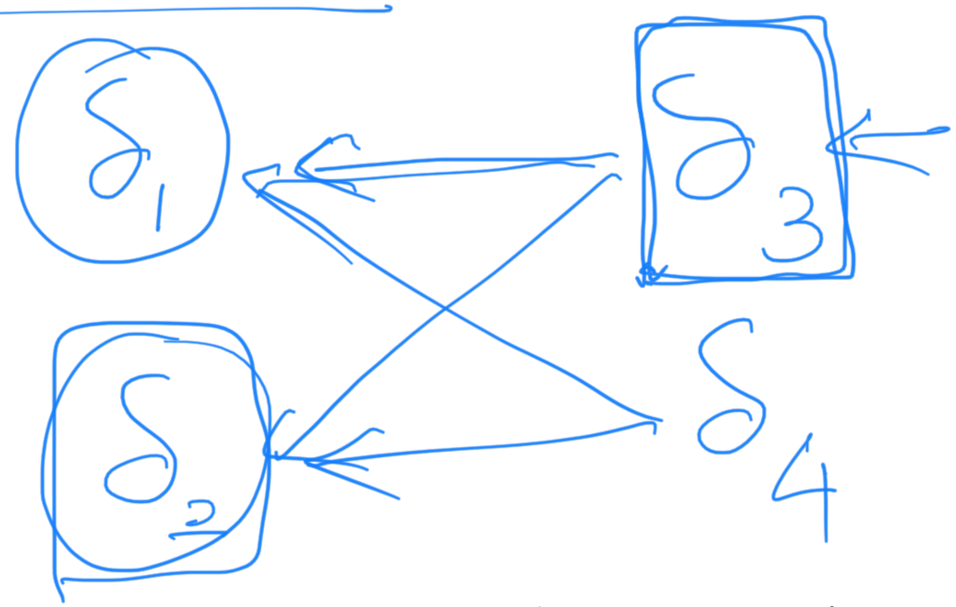
- Parameters: <$\boldsymbol{W}^{(1)}, \boldsymbol{W}^{(2)}, \ldots$>
- Objective:
  - **Error minimization**
    - **Backpropagation (aka backprop)**

Backpropagation Algorithm

- Two phases:
  - Forward phase: compute output z_j for each node j



  - Backward phase: compute output δ_j for each node j



Forward Phase

- Propagate inputs forward through the network to compute the output of each node
- Output z_j at node j
  - $z_j = h(a_j)$ where $\boxed{a_j = \sum_i w_{ji} z_i}$

Backward Phase

- Use chain rule to recursively compute gradient
  - For each weight w_{ji}

$$\delta_{ji} = \frac{\partial E_n}{\partial w_{ji}} = \frac{\partial E_n}{\partial a_j}\frac{\partial a_j}{\partial w_{ji}}$$

  - Let $\delta_j = \dfrac{\partial E_n}{\partial a_j}$
  - Then

$$\delta_j = \begin{cases} h'(a_j)(z_j - y_j) & base\ case:\ j \in output\ nodes \\ h'(a_j)\sum_k w_{kj}\delta_k & recursion:\ j \in hidden\ nodes \end{cases}$$

  - Since

$$a_j = \sum_k w_{ji}z_i \quad then \quad \frac{\partial a_j}{\partial w_{ji}} = z_i$$

  - Therefore, $\delta_{ji} = \dfrac{\partial E_n}{\partial w_{ji}} = \dfrac{\partial E_n}{\partial a_j}\dfrac{\partial a_j}{\partial w_{ji}} = \delta_j z_i$

An Example

- A simple two-layer network:
  - Hidden nodes (Tanh): $h(a) = tanh(a) = \dfrac{e^a - e^{-a}}{e^a + e^{-a}}$
    - Note: $\cancel{\boxed{tanh'(a) = (1 - (tanh'(a))^2}} = (1 - tanh(a)^2)$
  - Output nodes (Identity): $\boxed{h(a) = a}$
- Objective function: squared error

The derivation

- Forward phase
  - Hidden nodes: $a_j = \sum_i w_{ji} x_i$ $\qquad z_j = \tanh(a_j)$
  - Output nodes: $a_k = \sum_j w_{kj} z_j$ $\qquad z_k = a_k$
- Backward phase
  - Output nodes: $\boxed{\delta_k} = z_k - y_k$
  - Hidden nodes: $\delta_j = \left[1 - (z_j)^2\right] \sum_k w_{kj} \delta_k$
- Gradients
  - Hidden layer: $\dfrac{\partial E_n}{\partial w_{ji}} = \delta_j x_i = \left[1 - (z_j)^2\right] \sum_k w_{kj} \delta_k x_i$
  - Output layer: $\dfrac{\partial E_n}{\partial w_{kj}} = \boxed{\delta_k z_j} = (z_k - y_k) z_j$

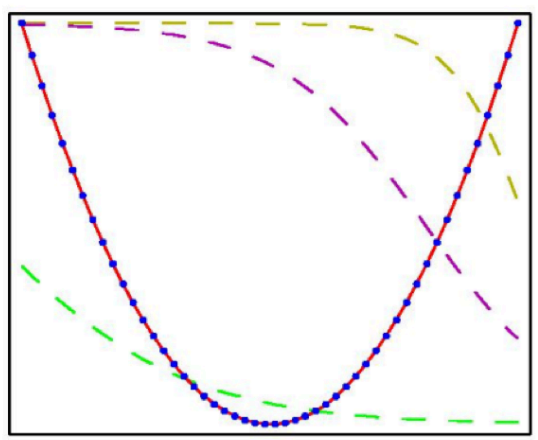
The derivation

- Forward phase
  - Hidden nodes: $a_j = \sum_i w_{ji} x_i$ $\qquad z_j = tanh(a_j)$
  - Output nodes: $a_k = \sum_j w_{kj} z_j$ $\qquad z_k = a_k$
- Backward phase
  - Output nodes: $\delta_k = z_k - y_k$
  - Hidden nodes: $\delta_j = [1 - (z_j)^2] \sum_k w_{kj} \delta_k$
- Gradients
  - Hidden layer: $\dfrac{\partial E_n}{\partial w_{ji}} = \delta_j x_i = [1 - (z_j)^2] \sum_k w_{kj} \delta_k x_i$
  - Output layer: $\dfrac{\partial E_n}{\partial w_{kj}} = \delta_k z_j = (z_k - y_k) z_j$

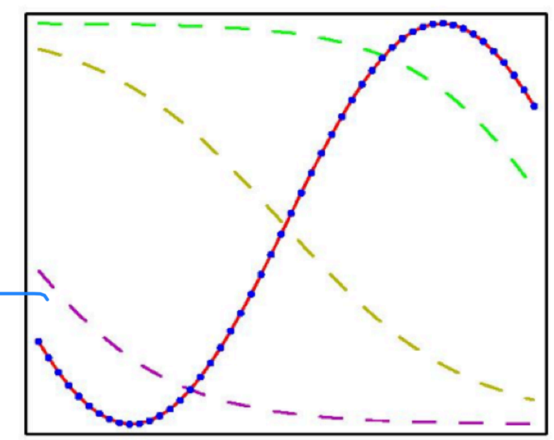
Non-linear regression examples

- Two layer network:
  - 3 tanh hidden units and 1 identity output unit

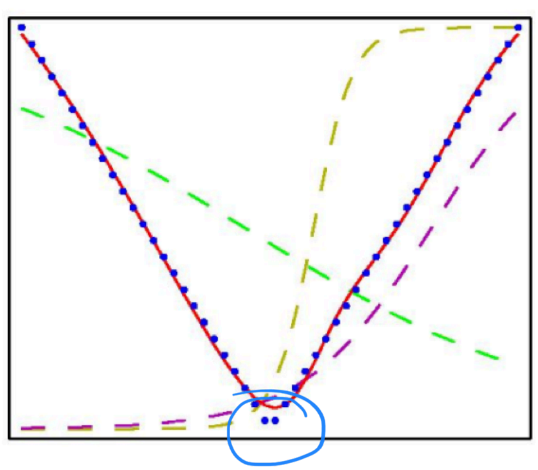
$y = x^2$



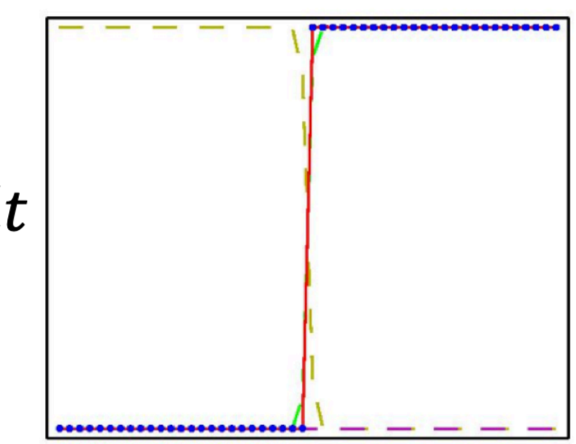
$y = \sin x$



$y = |x|$



$y = \int_{-\infty}^{x} \delta(t)dt$



Demo Time 🙂

https://playground.tensorflow.org/