CS 4824/ECE 4424: Linear Regression

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Linear regression

- Regression $p(y \mid x) = \mathcal{N}(f(x), \sigma)$

- Assume $f(x)$ is a linear function of $x$
  - $p(y \mid x) = \mathcal{N}(w_0 + w_1 x, \sigma)$
  - $\mathbb{E}(y \mid x) = w_0 + w_1 x$

- Note: to make our parameters explicit, let’s write
  - $W = \langle w_0, w_1 \rangle$
  - $p(y \mid x, W) = \mathcal{N}(w_0 + w_1 x, \sigma)$
Training linear regression

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Training linear regression

\[ p(y|x) = \mathcal{N}(f(x), \sigma) \]

- How can we learn \( W \) from data?

- Learn \( W \) using Maximum Conditional Likelihood Estimation!
  \[ W_{MCLE} = \arg\max_W \prod_l P(Y^l | X^l, W) \]
  \[ W_{MCLE} = \arg\max_W \sum_l \ln p(y^l | x^l, W) \]
  Where \( P(y|x, W) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{y-f(x, W)}{\sigma} \right)^2} \)
Training linear regression

- Learn $W$ using Maximum Conditional Likelihood Estimation:
  \[
  W_{MCLE} = \arg \min_W \sum_l (y - f(x, W))^2
  \]

- This corresponds to minimizing sum of squared errors

- How to perform the minimization in order to choose optimal
  \[
  W = < w_0, w_1 > ?
  \]
Gradient descent

\[ \nabla E(\overrightarrow{w}) = \left[ \frac{\partial E}{\partial w_0}, \frac{\partial E}{\partial w_1}, \ldots, \frac{\partial E}{\partial w_n} \right] \]

\[ \text{Training Rule: } \overrightarrow{w}^{(i+1)} \leftarrow \overrightarrow{w}^i - \eta \nabla E(\overrightarrow{w}) \]

\[ \Delta w_i = -\eta \frac{\partial E}{\partial w_i} \]
Minimizing squared error: gradient descent

\[
\frac{\partial E}{\partial w_1} = \sum_l 2(y^l - (w_o + w_1x^l))(-x^l)
\]

\[
= -2 \sum_l (y^l - (w_o + w_1x^l))(x^l)
\]

\[
\frac{\partial E}{\partial w_0} = -2 \sum_l (y^l - (w_o + w_1x^l))
\]
Minimizing squared error: gradient descent

\[
\frac{\partial E}{\partial w_1} = -2 \sum_l (y^l - (w_0 + w_1 x^l))(x^l) \quad \frac{\partial E}{\partial w_0} = -2 \sum_l (y^l - (w_0 + w_1 x^l))
\]

- Update rule:

\[
\begin{align*}
\omega_1 &\leftarrow \omega_1 - \eta \left[ -2 \sum_{l} (y^l - (w_0 + w_1 x^l))(x^l) \right] \\
\omega_0 &\leftarrow \omega_0 + 2\eta \sum_{l} (y^l - (w_0 + w_1 x^l)) \\
\end{align*}
\]
Linear regression more generally

\[
p(y | x) = \mathcal{N}(f(x), \sigma I) \quad \bar{X} = < x_1, x_2, \ldots, x_n >
\]

\[
f(x) = w_0 + \sum_{i=1}^{n} w_i x_i
\]

\[
p(y | x) = \mathcal{N}(w_0 + \sum_{i=1}^{n} w_i x_i, \sigma I)
\]

\[
\mathbb{E}(y | x) = w_0 + \sum_{i=1}^{n} w_i x_i
\]

\[
p(y | x, W) = \mathcal{N}(w_0 + \sum_{i=1}^{n} w_i x_i, \sigma I) \quad \bar{W} = < w_0, x_1, \ldots, x_n >
\]
Minimizing squared error more generally: gradient descent

- **Gradient descent algorithm**: iterate until change $< \varepsilon$
  
  - $\forall i$ repeat $\quad w_i \leftarrow w_i + 2\eta \sum_l X^l(Y^l - (w_0 + \sum_{j=1}^{n} w_j x_j))$
  
  - assume $X_0 = 1$ for $w_0$
How about MAP estimation?

\[ W_{MAP} = \arg\max_{W} (-c \sum_{i} w_i^2) + \sum_{l} \ln P(Y^l | X^l, W) \]

- Called a “regularization” term
- Helps reduce overfitting, especially for sparse data situations
- Keeps weights near zero with prior \( W \sim \mathcal{N}(0, \sigma I) \), or whatever the prior suggests
Demo Time 😊

https://lukaszkujawa.github.io/gradient-descent.html
Summary of regression

- Assuming $p(y | x, W) = \mathcal{N}(w_0 + w_1 x, \sigma)$

- MLE corresponds to minimizing sum of squared errors
- MAP estimate minimizes SSE plus sum of squared weights
- Again, learning is an optimization problem once we choose our objective function
  - maximize data likelihood
  - maximize posterior prob of $W$
- Again, we can use gradient descent as a general learning algorithm

- Be careful about outliers while performing regression

- **NOTE**: Almost nothing here required that $f(x)$ be linear in $x$