CS 4824/ECE 4424: Graphical Models I

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Graphical Models

- **Key Idea**:
  - Express sets of conditional independence assumptions via graph structure
  - Graph structure plus associated parameters define joint probability distribution over set of variables/ nodes

Conditional independence

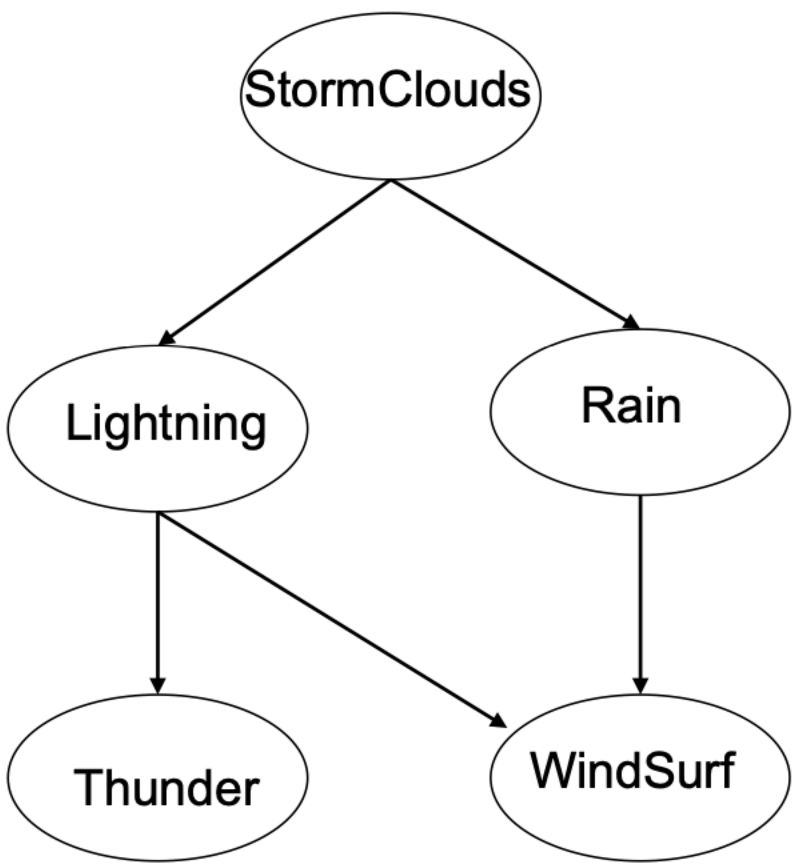
- X is conditionally independent of Y given Z, if the probability distribution governing X is independent of the value of Y given the value of Z
    - $(\forall i, j, k)\ P(X = x_i | Y = y_j, Z = z_k) = P(X = x_i | Z = z_k)$
    - Or equivalently, P(X|Y Z) = P(X|Z)
- P(Thunder|Rain, Lightning) = P(Thunder|Lightning)
    - That is, Thunder and Rain are conditionally independent

Marginal independence

- X is marginally independent of Y if
    - $(\forall i, j)\ P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$
- Equivalently if
    - $(\forall i, j)\ P(X = x_i \mid Y = y_j) = P(X = x_i)$
- Equivalently if
    - $(\forall i, j)\ P(Y = y_i \mid X = x_j) = P(Y = y_i)$

Bayesian network

- A directed acyclic graph defining a joint probability distribution over a set of variables
- Each node denotes a random variable
- A conditional probability distribution (CPD) is associated with each node N, defining P(N | Parents(N))



Parents	P(W|Pa)	P(¬W|Pa)
L, R	0	1.0
L, ¬R	0	1.0
¬L, R	0.2	0.8
¬L, ¬R	0.9	0.1

The joint distribution over all variables in the network is defined in terms of these CPD's, plus the graph

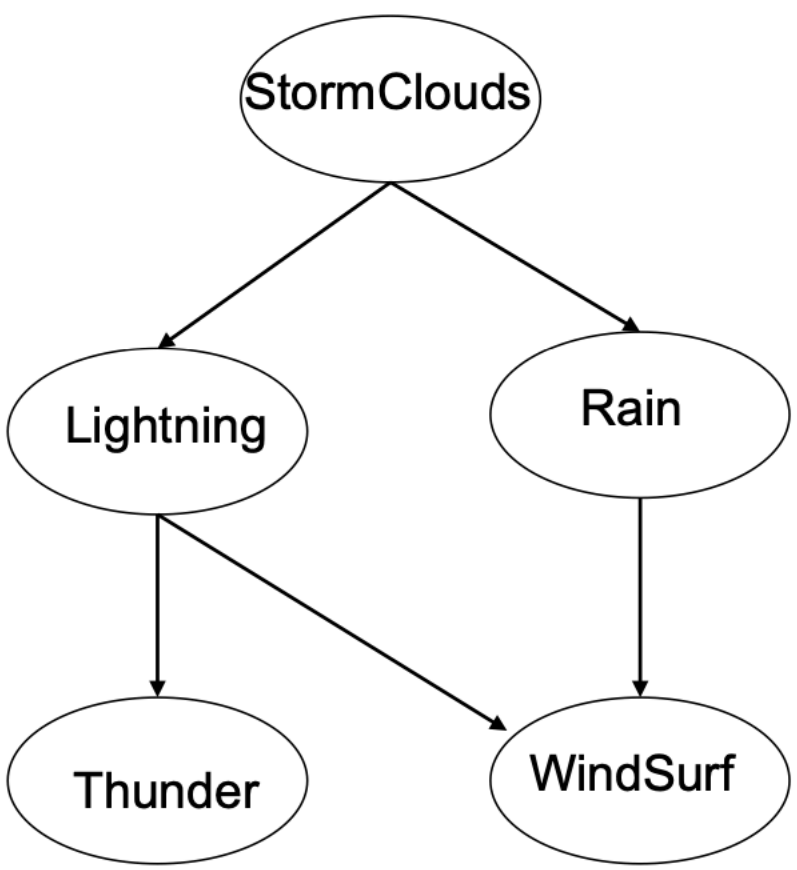
Bayesian network more formally

- A Bayes network represents the joint probability distribution over a collection of random variables

- A Bayes network is a directed acyclic graph and a set of CPD's
  - Each node denotes a random variable
  - Edges denote dependencies
  - CPD for each node X_i defines $P(X_i \mid Pa(X_i))$
  - The joint distribution over all variables is defined as
    $$P(X_1 \ldots X_n) = \prod_i P(X_i \mid Pa(X_i))$$

Bayesian network: conditional independence

- What can we say about conditional independence in a Bayes Net?
- Each node is conditionally independent of its non-descendents, given only its immediate parents



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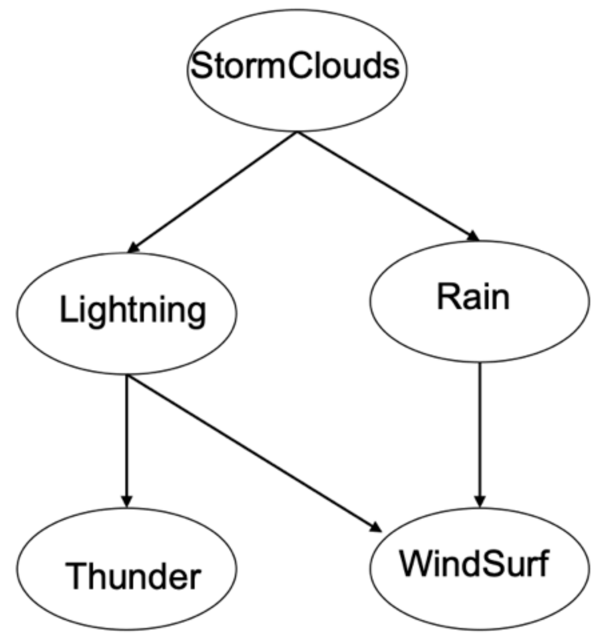
Bayesian network

- CPD for each node X_i describes $P(X_i \mid Pa(X_i))$



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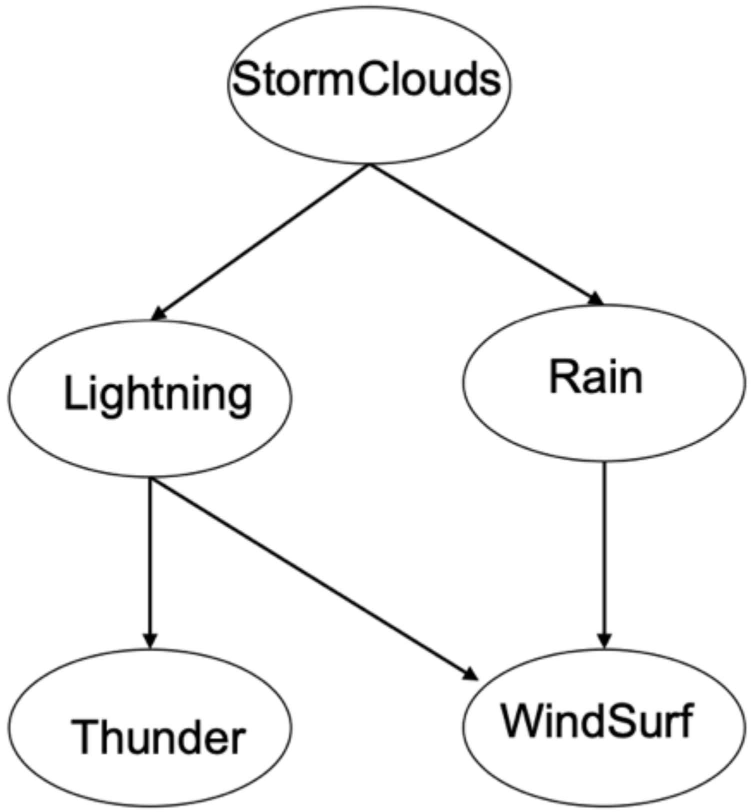
- Chain rule of probability
  $P(S, L, R, T, W) = P(S)P(L \mid S)P(R \mid S, L)P(T \mid S, L, R)P(W \mid S, L, R, T)$

- But in Bayes net: $P(X_1 \ldots X_n) = \prod_i P(X_i \mid Pa(X_i))$

How many parameters?



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- In full joint distribution?

- Given this Bayes net?

Algorithm for constructing Bayes network

- Choose an ordering over variables, e.g. $X_1, X_2, \ldots, X_n$
- For i=1 to n
  - Add X_i to the network
  - Select parents $Pa(X_i)$ as minimal subset of $X_1 \ldots X_{i-1}$ such that $P(X_i \mid Pa(X_i)) = P(X_i \mid X_1 \ldots X_{i-1})$

- Notice this choice of parents assures

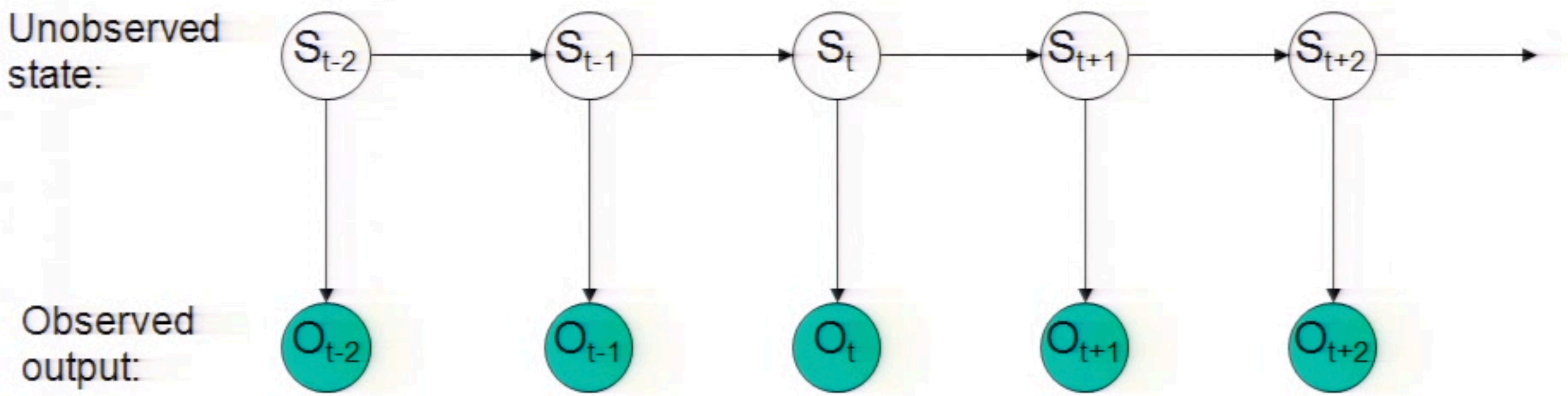
$$P(X_1 \ldots X_n) = \prod_i P(X_i \mid X_1 \ldots X_{i-1}) = \prod_i P(X_i \mid Pa(X_i))$$

What is the Bayes net for X_1, X_2,…,X_n with NO assumed conditional independence ?

What is the Bayes net for Naïve Bayes?

Hidden Markov Model

- Assume the future is conditionally independent of the past, given the present



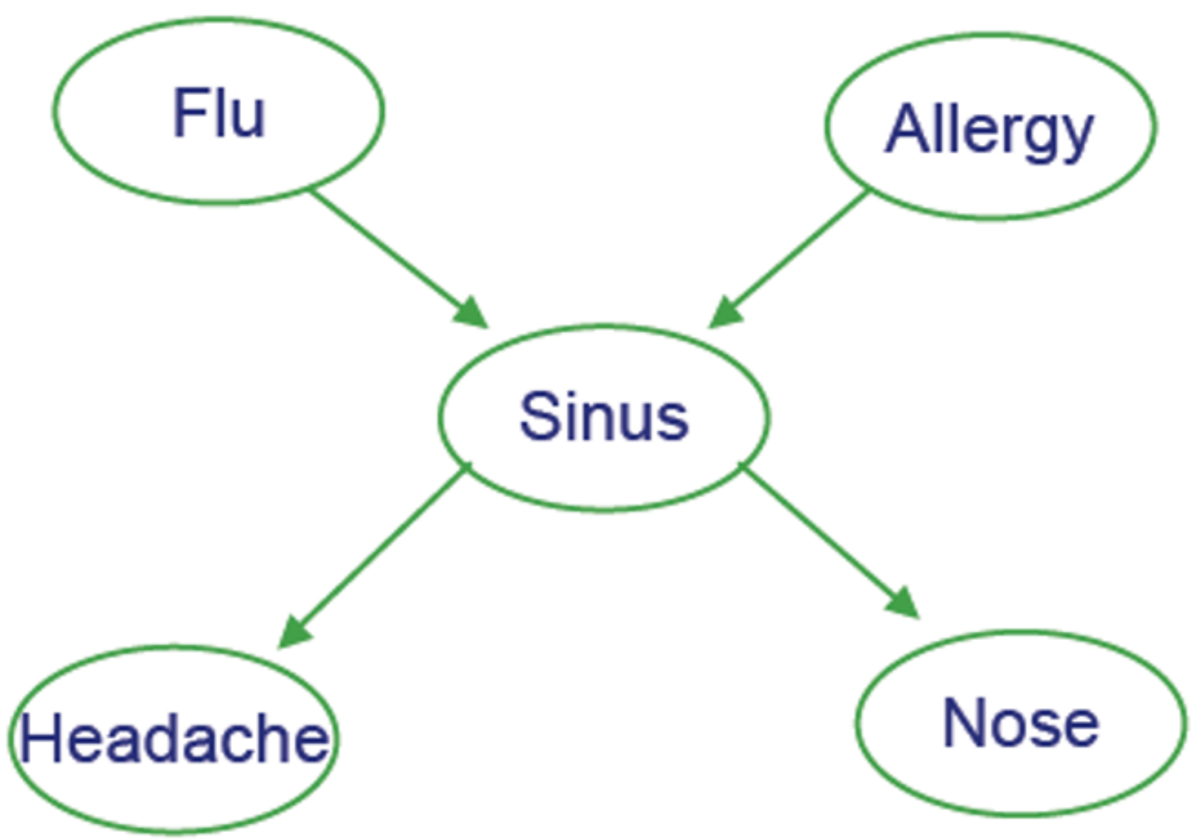
$$P(S_{t-2}, O_{t-2}, S_{t-1}, \ldots, O_{t+2}) =$$

Inference in Bayes Nets

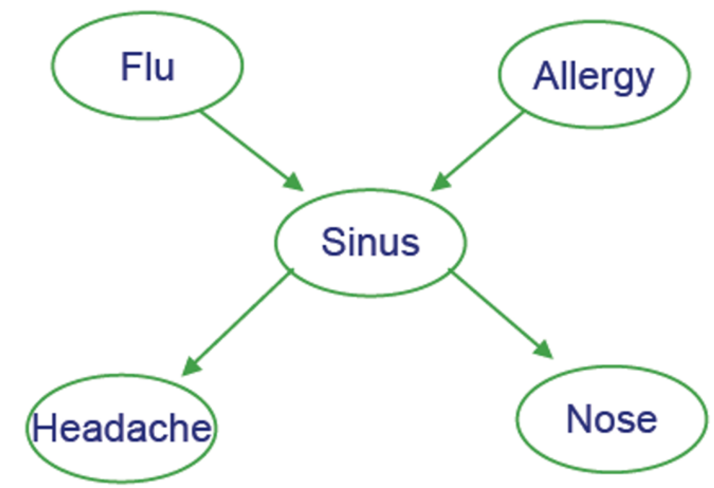
- In general, intractable (NP-complete)
- For certain cases, tractable
  - Assigning probability to fully observed set of variables
  - Or if just one variable unobserved
  - Or for singly connected graphs (ie., no undirected loops)
    - Belief propagation
- Sometimes use Monte Carlo methods
  - Generate many samples according to the Bayes Net distribution, then count up the results

Example

- Bird flu and Allegies both cause Sinus problems
- Sinus problems cause Headaches and runny Nose



Prob. of joint assignment: easy

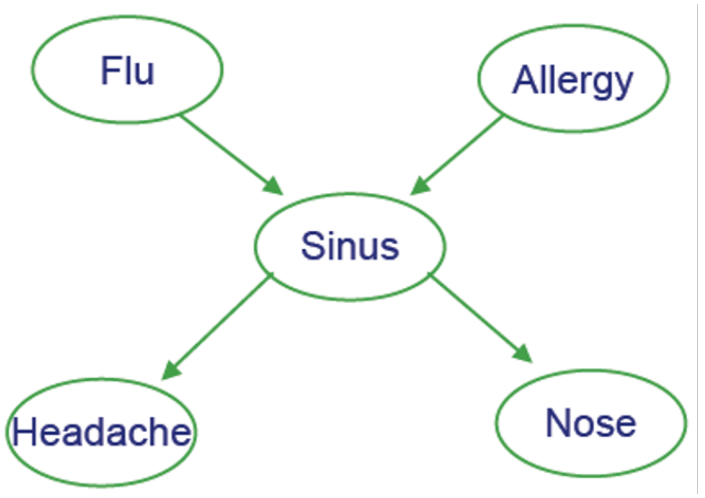


- Suppose we are interested in joint assignment <F=f,A=a,S=s,H=h,N=n>
- What is P(f,a,s,h,n)?

let's use P(a,b) as shorthand for P(A=a, B=b)

Prob. of marginals: not so easy

- How do we calculate P(N=n)?



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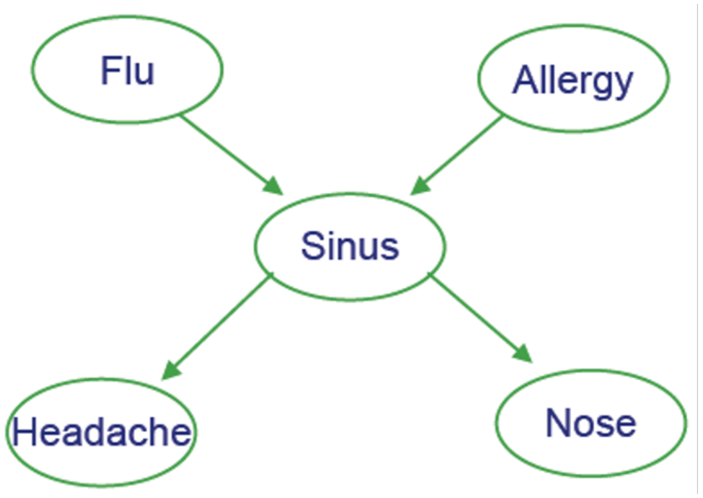
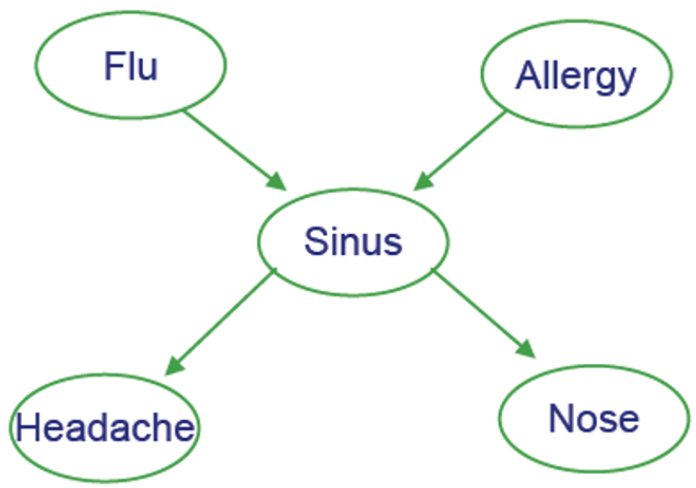
Generating a sample from joint distribution: easy

- How can we generate random samples drawn according to P(F,A,S,H,N)?



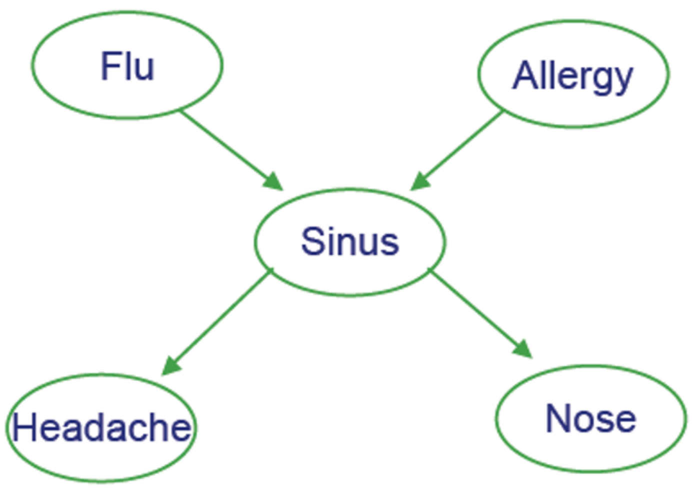
Generating a sample from joint distribution: easy



- How can we generate random samples drawn according to P(F,A,S,H,N)?
- random sample of F according to $P(F=1) = \theta_{F=1}$:
  - draw a value of r uniformly from [0,1]
  - if $r<\theta$ then output F=1, else F=0
- **Solution**:
  - draw a random value f for F, using its CPD
  - then draw values for A, for S|A,F, for H|S, for N|S

Generating a sample from joint distribution: easy

- Note we can estimate marginals like P(N=n) by generating many samples from joint distribution, then count the fraction of samples for which N=n



- Similarly, for anything else we care about P(F=1|H=1, N=0)
  - weak but general method

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