CS 4824/ECE 4424: Graphical Models II

Acknowledgement:
Many of these slides are derived from Tom Mitchell, Pascal Poupart, Pieter Abbeel, Eric Eaton, Carlos Guestrin, William Cohen, and Andrew Moore.

Bayesian network recap

- A Bayes network represents the joint probability distribution over a collection of random variables

- A Bayes network is a directed acyclic graph and a set of CPD's
  - Each node denotes a random variable
  - Edges denote dependencies
  - CPD for each node X_i defines $P(X_i \mid Pa(X_i))$
  - The joint distribution over all variables is defined as

$$P(X_1 \ldots X_n) = \prod_i P(X_i \mid Pa(X_i))$$

Bayesian network recap

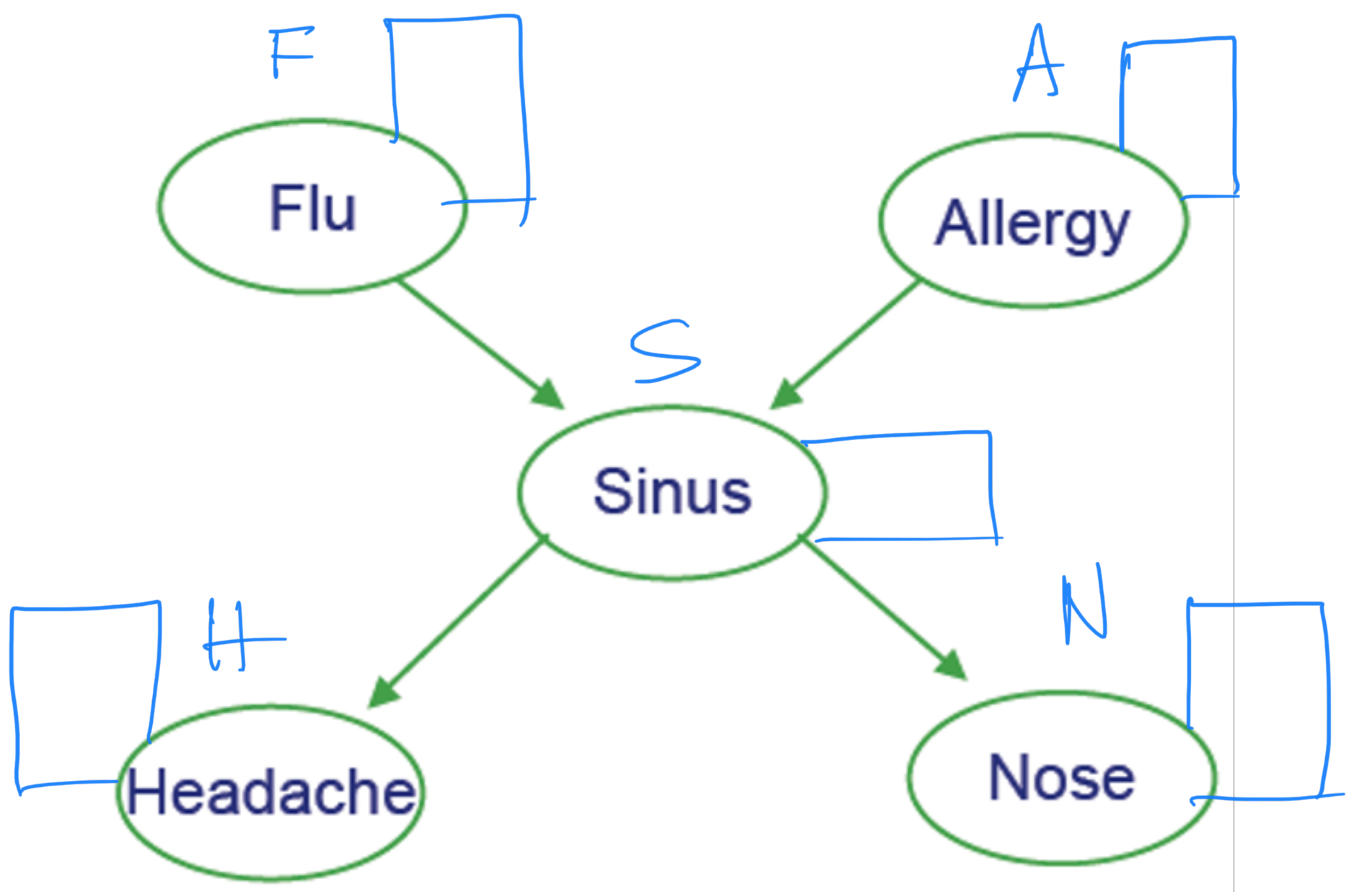
- Bayes nets are convenient representation for encoding dependencies / conditional independence
- BN = Graph plus parameters of CPD’s
    - Defines joint distribution over variables
    - Can calculate everything else from that
    - Though inference may be intractable

Inference in Bayes Nets

- In general, intractable (NP-complete)
- For certain cases, tractable
  - Assigning probability to fully observed set of variables
  - Or if just one variable unobserved
  - Or for singly connected graphs (ie., no undirected loops)
    - Belief propagation
- Sometimes use Monte Carlo methods
  - Generate many samples according to the Bayes Net distribution, then count up the results

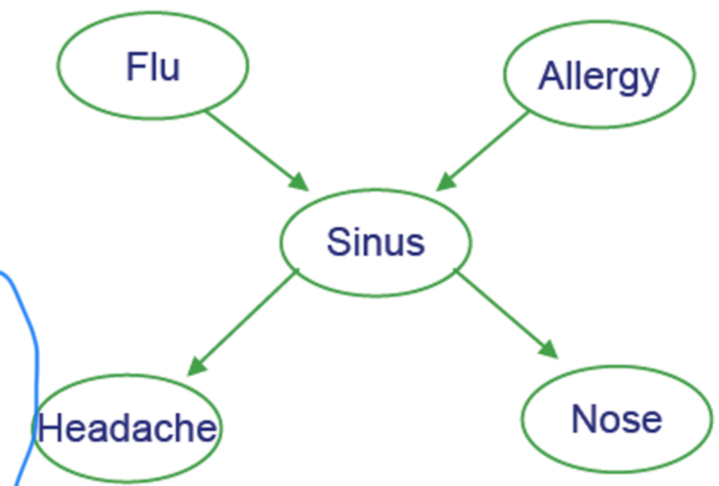
Example

- Bird flu and Allegies both cause Sinus problems
- Sinus problems cause Headaches and runny Nose



Prob. of joint assignment: easy

- Suppose we are interested in joint assignment <F=f,A=a,S=s,H=h,N=n>



S	00	01	10	11
0				
1				

$P(s \mid F, A)$

- What is P(f,a,s,h,n)?

$$P(F=f, A=a, S=s, H=h, N=n)$$
$$= P(F=f)\, P(A=a)\, \boxed{P(S=s \mid F=f, A=a)}\, P(H=h \mid S=s)\, P(N=n \mid S=s)$$

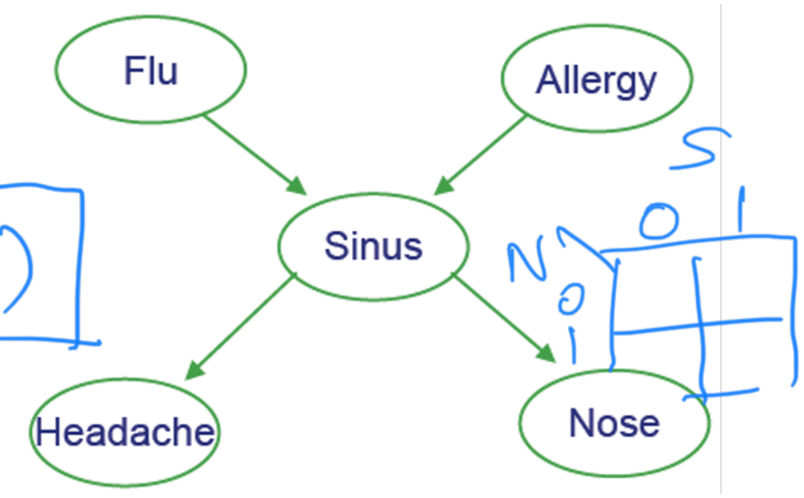
if we have K RV's, there will be K terms

Cost is linear in the number of RV's

let's use P(a,b) as shorthand for P(A=a, B=b)

Prob. of marginals: not so easy

- How do we calculate P(N=n)?



$$P(N=n) = \sum_{s} P(N=n \mid S=s)\, \boxed{P(S=s)}$$

chase up the Bayes Net

$$P(N=n) = \sum_{f,a,h,s} P(F=f, A=a, H=h, S=s, N=n)$$

Let's say we have K boolean RVs

How many terms do we have in the sum? 2^{K-1}

for each of them, we do K multiplications

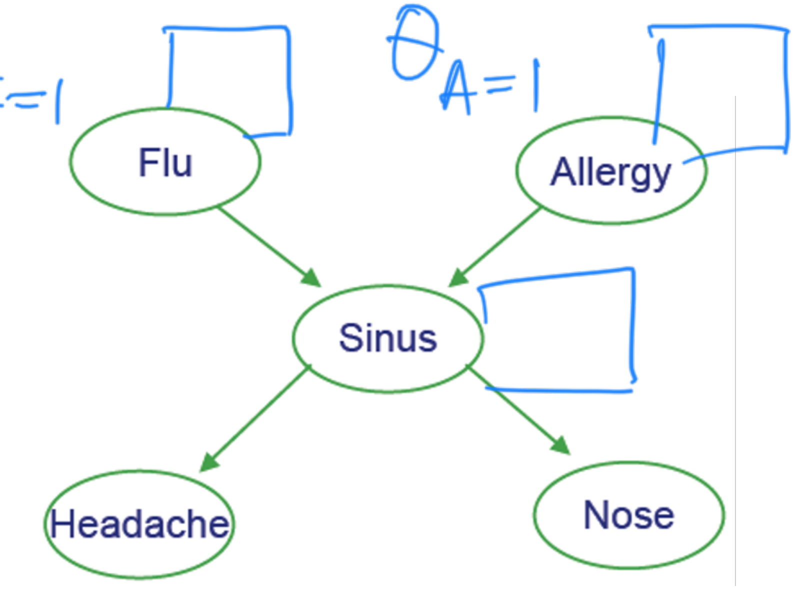
Cost = $\boxed{2^{K-1} \cdot K}$ computations!

let's use P(a,b) as shorthand for P(A=a, B=b)

Generating a sample from joint distribution: easy

- How can we generate random samples drawn according to P(F,A,S,H,N)?

$\theta_{F=1}$ $\theta_{A=1}$



$P(N=n)$? $\quad P(S=s \mid F=f, A=a)$

randomly draw a value for $F=f$

draw $r \in [0,1]$ uniformly randomly

if $r < \theta_{F=1}$

then output $f=1$

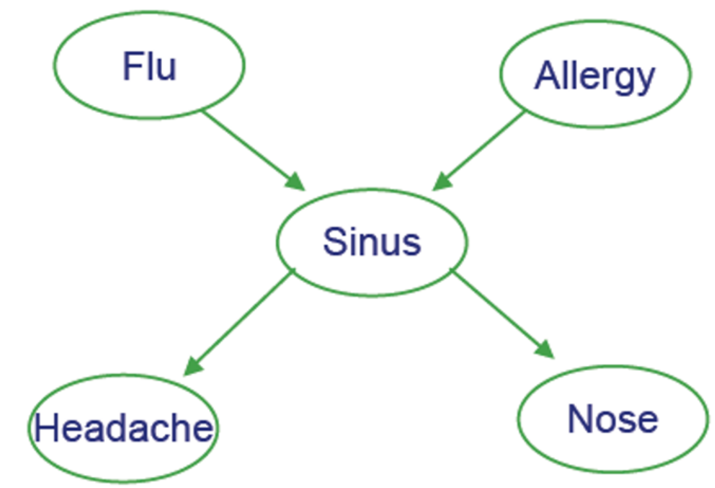
else $f=0$

Monte Carlo Sampling

Random Algorithm

N.B. with a fixed seed, your calculations are reproducible!

Generating a sample from joint distribution: easy



- How can we generate random samples drawn according to P(F,A,S,H,N)?
- random sample of F according to $P(F{=}1) = \theta_{F=1}$:
  - draw a value of r uniformly from [0,1]
  - if $r<\theta$ then output F=1, else F=0
- **Solution**:
  - draw a random value f for F, using its CPD
  - then draw values for A, for S | A,F, for H | S, for N | S

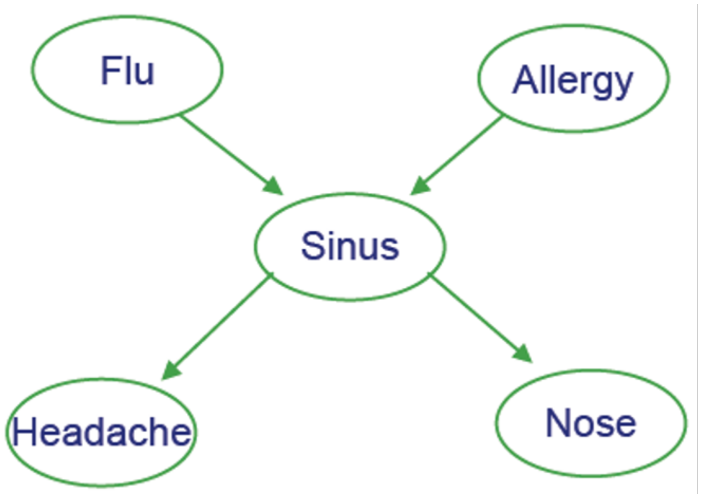
$\theta_{A=1} = 0.01$

$\theta_{F=1} = 0.9$

$P(S=1 \mid \boxed{A=1}, F=1)$

Caution: if prob. is very low (a.k.a. rare category) then you need to generate LOT of samples!!

Generating a sample from joint distribution: easy



- Note we can estimate marginals like P(N=n) by generating many samples from joint distribution, then count the fraction of samples for which N=n

- Similarly, for anything else we care about P(F=1 | H=1, N=0)

$$= \frac{P(F=1, H=1, N=0)}{P(H=1, N=0)}$$

  - weak but general method

Learning of Bayes Nets

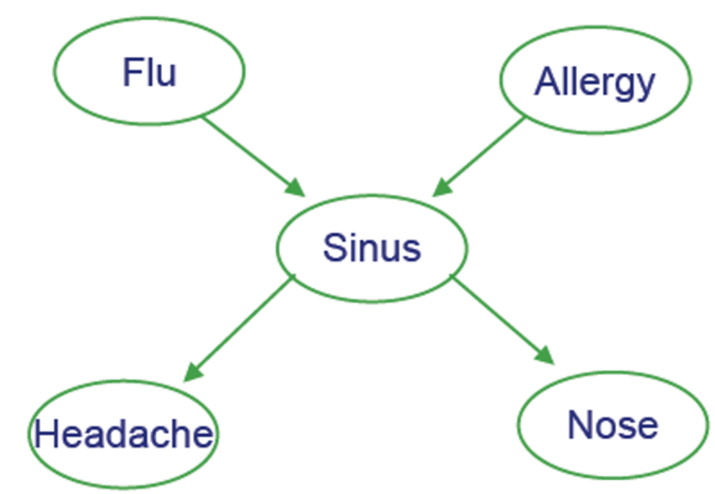
- Several types of of learning problems
  - Variable values may be fully observed / partly unobserved
- Easy case: learn parameters when data is *fully observed*
- Interesting case: graph *known*, data *partly known*

X∘ Gruesome case : graph structure is unknown, data partly known
"structure learning"

Learning CPTs from Fully Observed Data

- Example: Consider learning the parameter

$$\boxed{\theta_{\underset{S}{s}|\overset{F}{i}\overset{A}{j}} \equiv \boxed{P(S=1|F=i, A=j)}}$$



- Max Likelihood Estimate is

$$\boxed{\theta_{s|ij} = \frac{\sum_{k=1}^{K} \delta(f_k = i, a_k = j, s_k = 1)}{\sum_{k=1}^{K} \delta(f_k = i, a_k = j)}}$$

$$P(data|\theta) = \prod_{k=1}^{K} P(f_k, a_k, s_k, h_k, n_k)$$

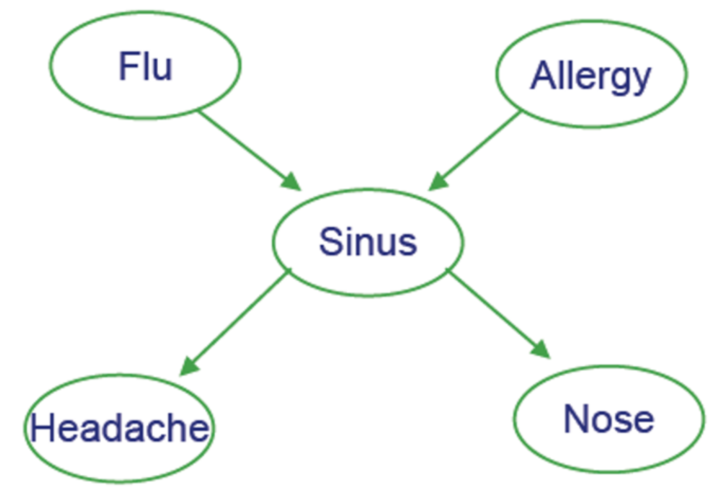
$\to$ k is the k^{th} training data.

$$= \prod_{k=1}^{K} P(f_k) P(a_k) P(S_k|f_k, a_k) P(h_k|S_k) P(n_k|S_k)$$

Learning CPTs from Fully Observed Data

- Example: Consider learning the parameter

$$\theta_{s|ij} \equiv P(S=1|F=i, A=j)$$



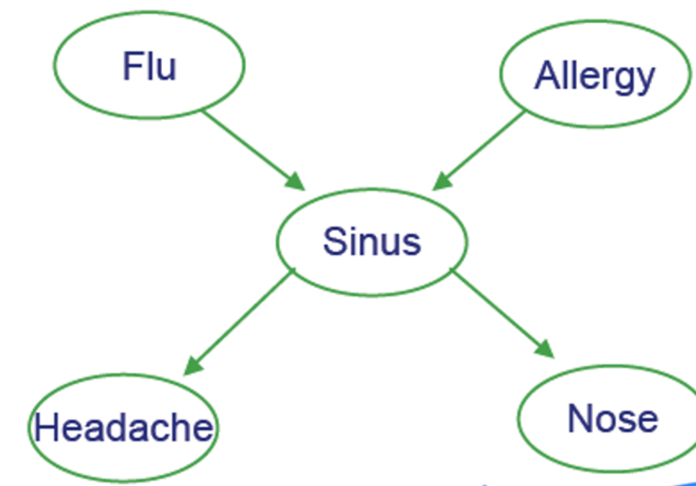
- Max Likelihood Estimate is

$$\theta_{s|ij} = \frac{\sum_{k=1}^{K} \delta(f_k = i, a_k = j, s_k = 1)}{\sum_{k=1}^{K} \delta(f_k = i, a_k = j)}$$

k^{th} training example

δ(x) = 1 if x=true,
= 0 if x=false

MLE estimate of $\theta_{s|ij}$ from fully observed data



- Maximum likelihood estimate

$$\theta \leftarrow \arg\max_{\theta} P(data|\theta)$$

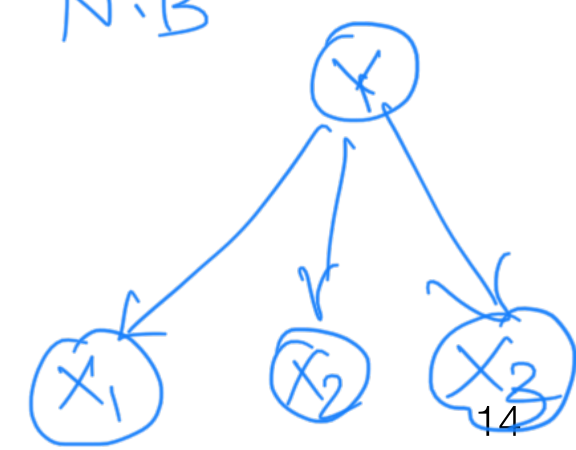
$$\theta \leftarrow \arg\max_{\theta} \log P(data|\theta)$$

- Our case

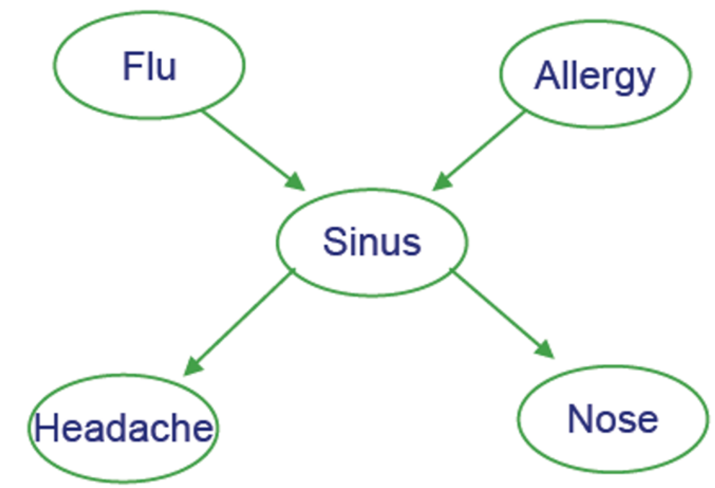
$$\log P(data|\theta) = \sum_{k=1}^{K} \boxed{\log P(f_k)} + \log P(a_k) + \overbrace{\log P(s_k|f_k, a_k)} + \log P(h_k|s_k) + \log P(n_k|s_k)$$

$$\frac{\partial \log P(data|\theta)}{\partial \boxed{\theta_{s|ij}}} = \sum_{k=1}^{K} \frac{\partial \log P(s_k|f_k, a_k)}{\partial \theta_{s|ij}}$$

N.B



MLE estimate of $\theta_{s|ij}$ from fully observed data



- Maximum likelihood estimate

$$\theta \leftarrow \arg\max_{\theta} \log P(data|\theta)$$

- Our case

$$P(data|\theta) = \prod_{k=1}^{K} P(f_k, a_k, s_k, h_k, n_k)$$

$$P(data|\theta) = \prod_{k=1}^{K} P(f_k)P(a_k)P(s_k|f_k a_k)P(h_k|s_k)P(n_k|s_k)$$

$$\log P(data|\theta) = \sum_{k=1}^{K} \log P(f_k) + \log P(a_k) + \log P(s_k|f_k a_k) + \log P(h_k|s_k) + \log P(n_k|s_k)$$

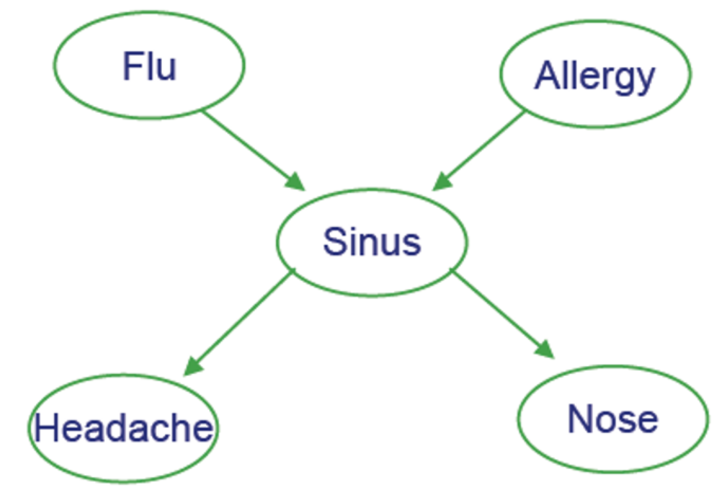
$$\frac{\partial \log P(data|\theta)}{\partial \theta_{s|ij}} = \sum_{k=1}^{K} \frac{\partial \log P(s_k|f_k a_k)}{\partial \theta_{s|ij}}$$

$$\theta_{s|ij} = \frac{\sum_{k=1}^{K} \delta(f_k = i, a_k = j, s_k = 1)}{\sum_{k=1}^{K} \delta(f_k = i, a_k = j)}$$

Estimate θ from partly observed data

- What if FAHN observed, but not S?
- Can't calculate MLE

$$\theta \leftarrow \arg\max_{\theta} \log \prod_{k} P(f_k, a_k, s_k, h_k, n_k | \theta)$$



- Let X be all *observed* variable values (over all examples)
- Let Z be all unobserved variable values
- Can't calculate MLE

$$\theta \leftarrow \arg\max_{\theta} \log P(X, Z | \theta)$$

- What to do?