CS 4824/ECE 4424: Logistic Regression

Acknowledgement:
Many of these slides are derived from Tom Mitchell, Pascal Poupart, Pieter Abbeel, Eric Eaton, Carlos Guestrin, William Cohen, and Andrew Moore.

Logistic Regression

Idea:

- Naïve Bayes allows estimating P(Y|X) by learning P(Y) and P(X|Y)
- Why not learn P(Y|X) directly?

Problem setting

- Consider learning $f: X \rightarrow Y$
  - X is a vector of real-valued features $<X_1, X_2, \ldots, X_n>$
  - Y is boolean
  - Assume all X_i's are conditionally independent given Y
  - Model $P(X_i | Y = y_k)$ as Gaussian ~ $\mathcal{N}(\mu_{ik}, \sigma_i)$
  - Model P(Y) as Bernoulli (π)

- Given that, what's the parametric form of $P(Y|X)$?

Parametric form of P(Y|X)

- $$P(Y=1|X) = \frac{P(Y=1)P(X|Y=1)}{P(Y=1)P(X|Y=1) + P(Y=0)P(X|Y=0)}$$

Parametric form of P(Y|X)

$$P(x|y_k) = \frac{1}{\sigma_{ik}\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{x-\mu_{ik}}{\sigma_{ik}}\right)^2\right)}$$

- $\sum_i \ln \frac{P(X_i|Y=0)}{P(X_i|Y=1)}$

Parametric form of P(Y|X)

- Therefore, $P(Y=1|X) = \frac{1}{1 + exp(w_0 + \sum_{i=1}^{n} w_i x_i)}$
  - where, $w_0 = \ln\frac{1-\pi}{\pi} + \sum_i \frac{\mu_{i1}^2 - \mu_{i0}^2}{2\sigma_i^2}$; and $w_i = \frac{\mu_{i0} - \mu_{i1}}{\sigma_i^2}$ for $i = 1 \ldots n$

Very convenient!

- $P(Y = 1 | X = < X_1, \ldots, X_n >) = \dfrac{1}{1 + exp(w_0 + \sum_{i=1}^{n} w_i x_i)}$
- implies
  - $P(Y = 0 | X = < X_1, \ldots, X_n >) =$
- implies
  - $\dfrac{P(Y = 0 | X)}{P(Y = 1 | X)} =$
- or equivalently
  - $\ln \dfrac{P(Y = 0 | X)}{P(Y = 1 | X)} =$

Very convenient!

- $P(Y=1 \mid X=<X_1,\ldots,X_n>) = \dfrac{1}{1+exp(w_0+\sum_{i=1}^{n} w_i x_i)}$
- implies
  - $P(Y=0 \mid X=<X_1,\ldots,X_n>) = \dfrac{exp(w_0+\sum_{i=1}^{n} w_i x_i)}{1+exp(w_0+\sum_{i=1}^{n} w_i x_i)}$
- implies
  - $$\frac{P(Y=0 \mid X)}{P(Y=1 \mid X)} = exp(w_0+\sum_{i=1}^{n} w_i x_i)$$

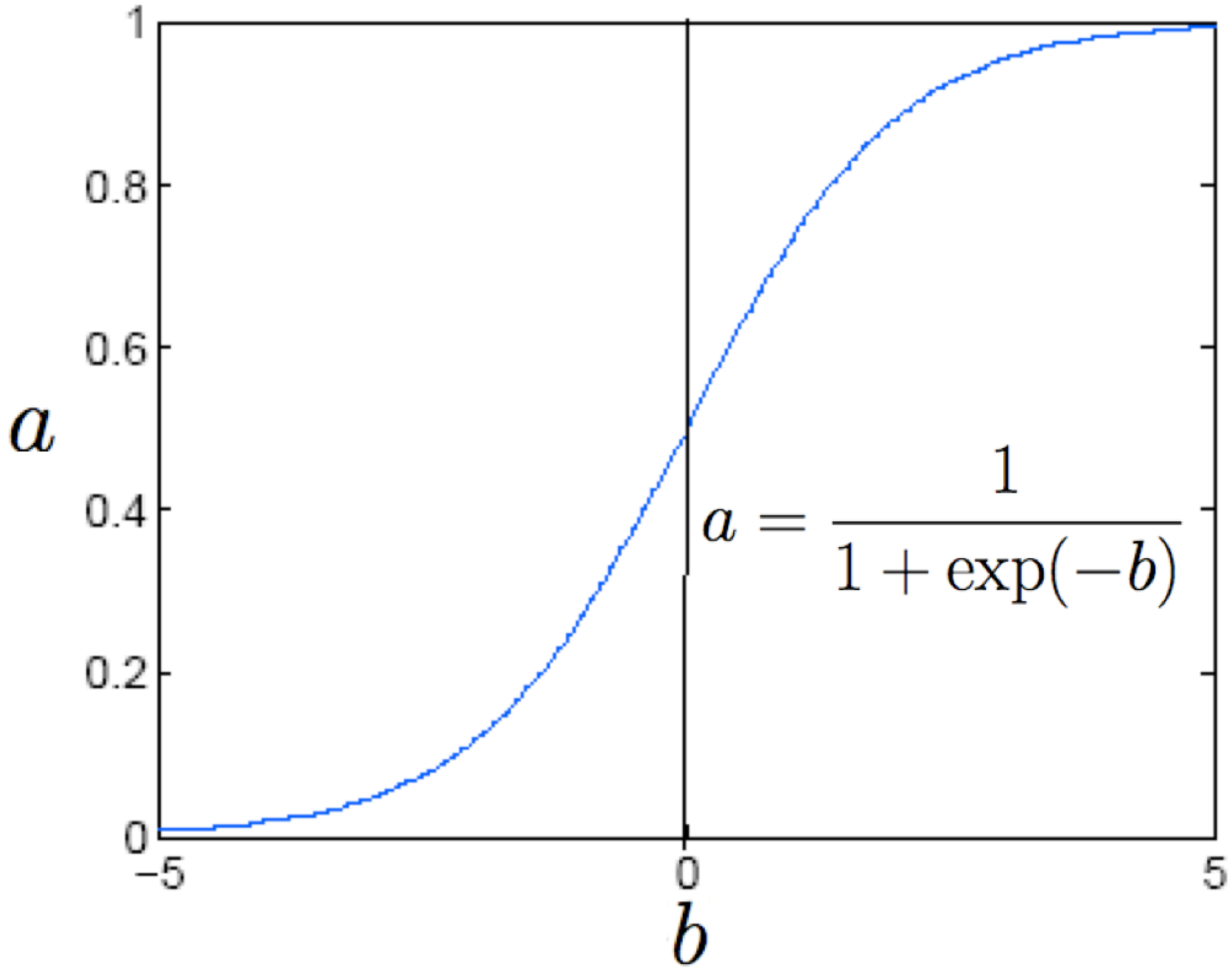
linear classification rule!

- or equivalently
  - $$\ln\frac{P(Y=0 \mid X)}{P(Y=1 \mid X)} = w_0+\sum_{i=1}^{n} w_i x_i$$

dot product of weights and the features

$$a\,.\,b = \sum_{i=1}^{n} a_i b_i$$

Logistic function



$$P(Y=1 \mid X) = \frac{1}{1 + exp(w_0 + \sum_{i=1}^{n} w_i x_i)}$$

Logistic regression more generally

- Logistic regression when Y not boolean, but still discrete valued
- Now $Y \in \{y_1, \ldots y_R\}$ and so we need to learn *R-1* sets of weights
    - for $k < R$: $$P(Y = y_k | X) = \frac{exp(w_{k0} + \sum_{i=1}^{n} w_{ki}x_i)}{1 + \sum_{j=1}^{R-1} exp(w_{j0} + \sum_{i=1}^{n} w_{ji}x_i)}$$
    - for $k = R$: $$P(Y = y_R | X) = \frac{1}{1 + \sum_{j=1}^{R-1} exp(w_{j0} + \sum_{i=1}^{n} w_{ji}x_i)}$$

Training logistic regression: MCLE

- We have L training examples $\{<X^1, Y^1>, \ldots, <X^L, Y^L>\}$
- Maximum likelihood estimate (MLE) for parameters W
  - $$W_{MLE} = \arg\max_W P(<X^1, Y^1> \ldots <X^L, Y^L> \mid W)$$
  - $$= \arg\max_W \prod_l P(<X^l, Y^l> \mid W)$$
- Maximum <u>conditional</u> likelihood estimate (MCLE)

Training logistic regression: MCLE

- We have L training examples {<X^1, Y^1>,…, <X^L, Y^L>}
- Maximum likelihood estimate (MLE) for parameters W
  - $$W_{MLE} = \arg\max_W P(<X^1, Y^1> \dots <X^L, Y^L> \mid W)$$
  - $$= \arg\max_W \prod_l P(<X^l, Y^l> \mid W)$$
- Maximum conditional likelihood estimate (MCLE)

Training logistic regression: MCLE

- We need to choose W = <w_0,…,w_n> to maximize the conditional likelihood of training data
    - where $P(Y=0|X,W) = \frac{1}{1 + exp(w_0 + \sum_{i=1}^{n} w_i x_i)}$
    - and $P(Y=1|X,W) = \frac{exp(w_0 + \sum_{i=1}^{n} w_i x_i)}{1 + exp(w_0 + \sum_{i=1}^{n} w_i x_i)}$
- Training data D = {<X^1, Y^1>,…, <X^L, Y^L>}
- Data likelihood is $\prod_l P(<X^l, Y^l> | W)$
- Data conditional likelihood is $\prod_l P(Y^l | X^l, W)$
- Therefore we need to estimate $W_{MCLE} = \arg\max_W \prod_l P(Y^l | X^l, W)$

Expressing conditional log likelihood

- $$l(W) = \ln \prod_l P(Y^l | X^l, W) = \sum_l \ln P(Y^l | X^l, W)$$
  - where $$P(Y=0|X,W) = \frac{1}{1 + exp(w_0 + \sum_{i=1}^n w_i x_i)}$$
  - and $$P(Y=1|X,W) = \frac{exp(w_0 + \sum_{i=1}^n w_i x_i)}{1 + exp(w_0 + \sum_{i=1}^n w_i x_i)}$$
- $$l(W) = \sum_l Y^l \ln P(Y^l = 1 | X^l, W) + (1 - Y^l) \ln P(Y^l = 0 | X^l, W)$$
  $$= \sum_l Y^l \ln \frac{P(Y^l = 1 | X^l, W)}{P(Y^l = 0 | X^l, W)} + \ln P(Y^l = 0 | X^l, W)$$
  $$= \sum_l Y^l (w_0 + \sum_i^n w_i X_i^l) - \ln(1 + exp(w_0 + \sum_i^n w_i X_i^l))$$

Maximizing conditional log likelihood

- $l(W) = \ln \prod_l P(Y^l | X^l, W)$

$$= \sum_l Y^l(w_0 + \sum_i^n w_i X_i^l) - \ln(1 + exp(w_0 + \sum_i^n w_i X_i^l))$$

- **Good news**: *l(W)* is a concave function of *W*
- **Bad news**: no closed-form solution to maximize *l(W)*

What do we do?