Section 4.1
The Basics of Counting

THE RULE OF SUM

If A and B are <u>disjoint</u> sets then $|A \cup B| = |A| + |B|$

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Example:

Suppose statement labels in a programming language must be a single letter or a single decimal digit.

Since a label cannot be both at the same time,

the number of labels

= the number of letters + the number of decimal digits

= 26 + 10 = 36.

THE RULE OF PRODUCT

$$| A \times B | = | A | | B|$$

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Example:

- Statement labels in Basic can be either

    - a single letter or

    - a letter followed by a digit.

Find the number of possible labels.

We can partition the set of all labels L into the disjoint subsets consisting of

- the set of single letter labels S

and

- the set of single letters followed by a digit D

and

- $L = S \quad D$.

Use the <u>rule of sum</u> to compute the cardinality of L if we can compute the cardinality of D.

- The elements of D are ordered pairs of the form [a, d] where a is an alphabetic character and d is a digit.

- By the <u>rule of product</u> the cardinality of D is the product of the cardinality of the two sets:

- (the alphabetic characters)(the decimal digits)

    $= (26)(10)$

= 260.

The cardinality of *L* is 26 + 260 = 286.

THE PRINCIPLE OF INCLUSION-EXCLUSION

If A and B are <u>not</u> disjoint:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

<u>Don't count objects in the intersection of two sets more than once!</u>

Example:

Count the number of bit strings of length 4 which begin with a 1 or end with a 00.

The set can be expressed as the union of

- the subset S of strings which begin with 1

and

- the subset O that end in 00.

Unfortunately the two subsets overlap.

- The cardinality of S is 8 (why?)

- The cardinality of O is 4 (why?).

Hence, by the exclusion-inclusion principle, the cardinality of the union is 12 minus the cardinality of the intersection.

How many strings are in the intersection?

Those strings that begin with 1 <u>and</u> end in 00 or 2 such strings.

The total number is 10 = 8 + 4 - 2.

Check:

• Strings in S that begin with 1:

1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111

• Strings O that end with 00:

0000, 0100, 1000, 1100

• 1000 and 1100 appear in both sets.

Count them once.

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More Counting Examples:

Find the number of three-letter initials where none of the letters is repeated.

Apply the rule of product remembering that a letter cannot appear twice to get

(26)(25)(24).

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Count the number of bit strings of length 4.

Apply the rule of product to get 2^4.

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Count the number of bit strings of length 4 or less.

Apply the rule of sum to get the disjoint subsets of length 1, 2, 3 and 4.

Then apply the rule of product to count each subset to get

$$2 + 4 + 8 + 16 = 2^1 + 2^2 + 2^3 + 2^4.$$

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Count the set S of 3 digit numbers which begin or end with an even digit.

Assume that 0 is even but a number cannot begin with 0.

The set is the union of the two subsets:

• The set B of three digit numbers that begin with 2, 4, 6 or 8.

This set has cardinality

$$(4)(10)(10).$$

(why?)

• The set C of three digit numbers that end with 0, 2, 4, 6, or 8 and do not begin with 0.

This set has cardinality

$$(5)(9)(10).$$

(why?)

• Now we use the inclusion-exclusion principle to eliminate the overlap of sets B and C.

Their intersection:

The 3 digit numbers that begin with 2, 4, 6, or 8 and end with 0, 2, 4, 6, or 8.

The intersection has the cardinality

$$(4)(10)(5)$$

Hence the cardinality is

$$400 + 450 - 200 = 650.$$
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