Section 4.3
Permutations and Combinations

Urn models

• We are given set of n objects in an urn (don't ask why it's called an "urn" - probably due to some statistician years ago) .

We are going to pick (select) r objects from the urn in sequence. After we choose an object

- we can replace it-(*selection with replacement*)
- or not -(*selection without replacement*).

If we choose r objects, how many different possible sequences of r objects are there?

Does the order of the objects matter or not?

Permutations

Selection without replacement of r objects from the urn with n objects.

A *permutation* is an arrangement.

Order matters.

After selecting the objects, two different orderings or arrangements constitute different permutations.

• Choose the first object n ways,

• Choose the second object (since selection is without replacement) (n - 1) ways,

.
.
.

• the rth object (n - r + 1) ways.

By the rule of product,

The number of permutations of n things taken r at a time

$$P(n,r) = n(n - 1)(n - 2) \dots (n - r + 1)$$

Note:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Example:

Let A and B be finite sets and let $| A | \leq | B |$.

Count the number of injections from A to B.

Note there are no injections if $| A | > | B |$ (why?)

There are P(| B |, | A |) injections:

We order the elements of A, {a1, a2, . . . } and assume the urn contains the set B.

- There are | B | ways to choose the image of a1,

- | B | - 1 ways to choose the image of a2,

and so forth.

Selection is without replacement. Otherwise we do not construct an injection.

Combinations

Selection is without replacement but

order does not matter.

It is equivalent to selecting subsets of size r from a set of size n.

Divide out the number of arrangements or permutations of r objects from the set of permutations of n objects taken r at a time:

The number of combinations of n things taken r at a time

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!\,r!}$$

Other names for C(n, r):

- *n choose r*
- *The binomial coefficient*

Example:

How many subsets of size r can be constructed from a set of n objects?

The answer is clearly C(n, r) since once we select the objects (without replacement) the order doesn't matter.

Corollary:

$$\sum_{r=0}^{n} C(n,r) = 2^n$$

Proof:

If we count the number of subsets of a set of size n, we get the cardinality of the power set.

Example:

Suppose you flip a fair coin n times. How many different ways can you get

- no heads? $C(n, 0)$
- exactly one head? $C(n, 1)$
- exactly two heads? $C(n, 2)$
- exactly r heads? $C(n, r)$
- at least 2 heads? $2^n - C(n, 0) - C(n, 1)$

Pascal's Identity:

$$C(n+1,k) = C(n,k-1) + C(n,k)$$

Proof:

We construct subsets of size k from a set with n + 1 elements given the subsets of size k and k-1 from a set with n elements.

The total will include

- all of the subsets from the set of size n which do not contain the new element

$$C(n, k),$$

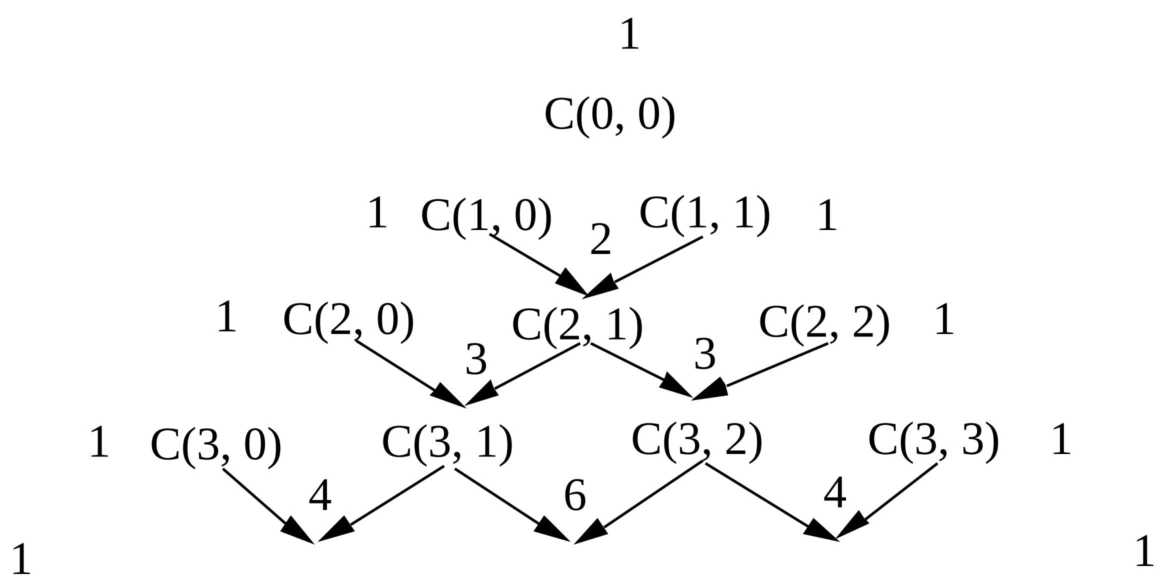
plus

• the subsets of size k - 1 with the new element added

C(n, k-1).

It produces

Pascal's triangle



A good way to evaluate C(n, r) for large n and r (to avoid overflow).

Example:

How many bit strings of length 4 have exactly 2 ones (or exactly 2 zeros)?

Analysis:

We solve the problem by determining the positions of the two ones in the bit string.

- place the first one - 4 possibilities
- place the second one - 3 possibilities

Hence it appears that we have (4)(3)= 12 possibilities.

We enumerate them to make sure:

0011, 0101, 1001, 0110, 1010, 1100.

There are actually only 6 possibilities. What is wrong?

The answer would be correct if we had two different objects to place in the string.

For example, if we were going to place an 'a' and a 'b' in the string we would have

00ab, 00ba, 0a0b, 0b0a, a00b, b00a,

and so forth for a total of 12.

But......the objects (1 and 1) are the same so the order is not important!

Divide through by the number of orderings = 2! = 2.

Therefore the answer is 12/2 = 6.

Example:

How many bit strings of length 4 have at least 2 ones?

Analysis:

Total the number of strings that have

- zero 1's = 1
- one 1 = 4

Total = $2^4 - 5 = 11$.

If the universe is the bit strings of length 4, what is the complement of the above set?

What is its cardinality?