Section 6.1
Relations and Their Properties

Definition: A *binary relation* R from a set A to a set B is a subset $R \quad A \times B$.

Note: there are no constraints on relations as there are on functions.

We have a common graphical representation of relations:

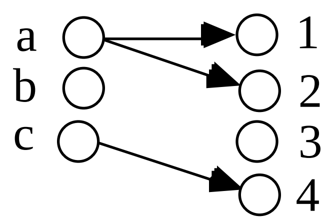
Definition: A *Directed graph* or a *Digraph* D from A to B is a collection of *vertices* $V \quad A \quad B$ and a collection of *edges* $R \quad A \times B$. If there is an ordered pair e = <x, y> in *R* then there is an *arc* or *edge* from x to y in D. The elements x and y are called the *initial* and *terminal* vertices of the edge e.

Examples:

- Let A = { a, b, c}
- B = {1, 2, 3, 4}
- R is defined by the ordered pairs or edges

{<a, 1>, <a, 2>, <c, 4>}

can be represented by the digraph D:

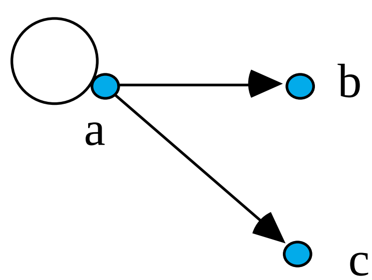


Definition: A binary relation R *on a set A* is a subset of $A \times A$ or a relation from A to A.

Example:

- A = {a, b, c}
- R = {<a, a>, <a, b>, <a, c>}.

Then a digraph representation of R is:



Note: An arc of the form <x, x> on a digraph is called a *loop*.

Question: How many binary relations are there on a set A?

Special Properties of Binary Relations

Given:

- A Universe U
- A binary relation R on a subset A of U

Definition: R is *reflexive* iff

$$x[x \quad U \rightarrow <x,x> \quad R]$$

Note: if U =   then the implication is true vacuously

The void relation on a void Universe is reflexive!

Note: If U is not void then <u>all</u> vertices in a reflexive relation must have loops!

Definition: R is *symmetric* iff

$$x \quad y[<x,y> \quad R \rightarrow <y,x> \quad R]$$

Note: <u>If</u> there is an arc <x, y> there must be an arc <y, x>.

Definition: R is *antisymmetric* iff

$$x \quad y[<x,y> \quad R \quad <y,x> \quad R \rightarrow x=y]$$

Note: If there is an arc from x to y there cannot be one from y to x if x≠y.

You should be able to show that logically: if <x, y> is in R and x ≠ y then <y, x> is not in R.

Definition: R is *transitive* iff

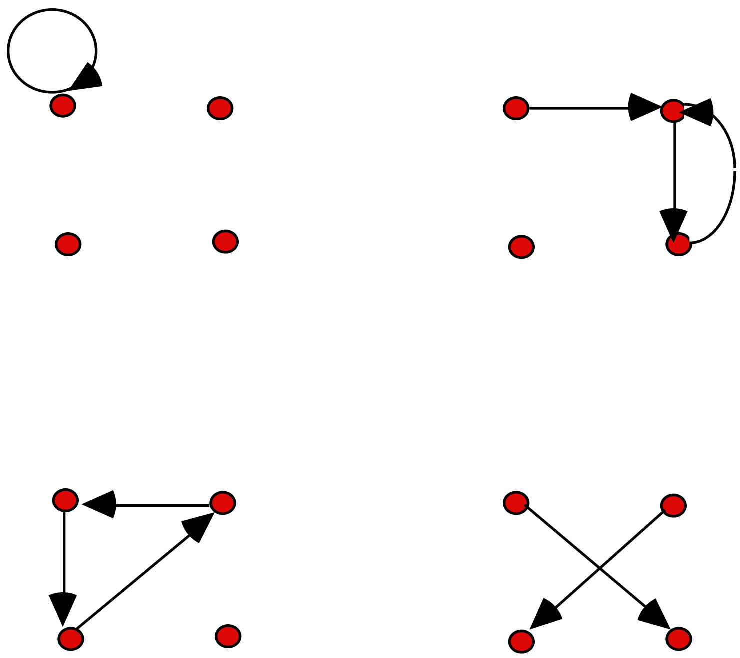
$$x \quad y \quad z[<x,y> \quad R \quad <y,z> \quad R \rightarrow <x,z> \quad R]$$

Note: if there is an arc from x to y and one from y to z then there must be one from x to z.

This is the most difficult one to check. We will develop algorithms to check this later.

Examples:

A. B.



C. D.

A: not reflexive
symmetric
antisymmetric
transitive

B: not reflexive
not symmetric
not antisymmetric
not transitive

C: not reflexive
not symmetric
antisymmetric
not transitive

D: not reflexive
not symmetric
antisymmetric
transitive

Combining Relations

Set operations

A very large set of potential questions -

Let R1 and R2 be binary relations on a set A:

If R1 has property 1

and

R2 has property 2,

does

R1 * R2 have property 3

where * represents an arbitrary binary set operation?

Example:

If

- R1 is symmetric,

and

- R2 is antisymmetric,

does it follow that

- R1   R2 is transitive?

If so, prove it. Otherwise find a counterexample.

Example:

Let R1 and R2 be transitive on A. Does it follow that

R1   R2

is transitive?

Consider

- A = {1, 2}
- R1 = {<1,2>}
- R2 = {<2, 1>}

Then R1   R2 = {<1, 2>, <2,1>} which is <u>not</u> transitive! (Why?)

Composition

Definition: Suppose

- R1 is a relation from A to B
- R2 is a relation from B to C.

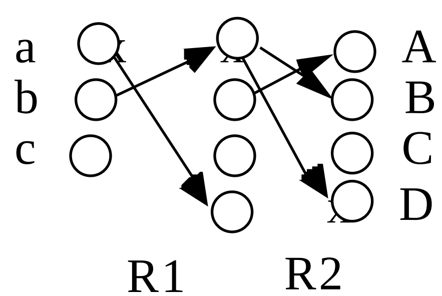
Then the composition of R2 with R1, denoted R2∘R1 is the relation from A to C:

If <x. y> is a member of R1 and <y, z> is a member of R2 then <x, z> is a member of R2∘R1.

Note: For <x, z> to be in the composite relation R2∘R1 there must exist a y in B . . . .

Note: We read them right to left as in functions.

Example:



R2∘R1 = {<b, D>, <b, B>}

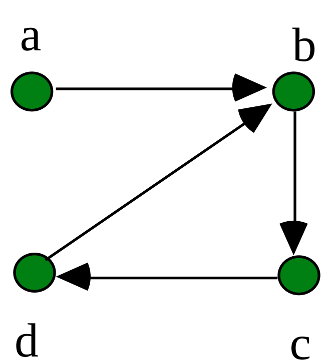
Definition: Let R be a binary relation on A. Then

Basis: $R^1 = R$

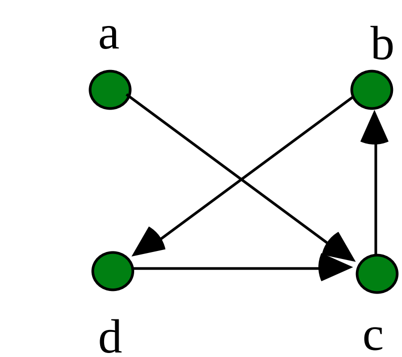
Induction: $R^{n+1} = R^n \circ R$

Note: an ordered pair <x, y> is in R^n iff there is a *path* of length n from x to y following the arcs (in the direction of the arrows) of R.

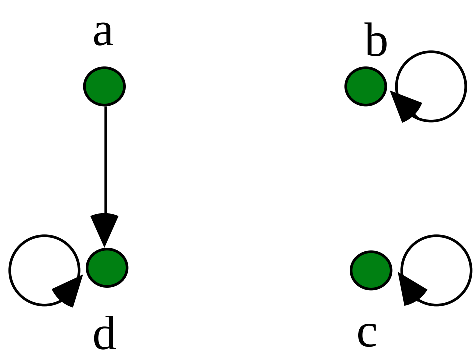
Example:



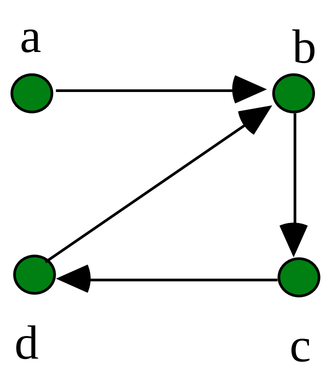
R



R^2



R^3



R^4

Very Important
Theorem:

R is transitive iff $R^n \quad$ R for n > 0.

Proof:

1. R transitive $\rightarrow R^n \quad R$

Use a direct proof and a proof by induction:

• Assume R is transitive.

• Now show $R^n \quad R$ by induction.

Basis: Obviously true for n = 1.

Induction:

• The induction hypothesis:

'assume true for n'.

• Show it must be true for n + 1.

$R^{n+1} = R^n \circ R$ so if <x, y> is in R^{n+1} then there is a z such that <x, z> is in R^n and <z, y> is in R.

But since $R^n \quad R$, <x, z> is in R.

R is transitive so <x, y> is in R.

Since <x, y> was an arbitrary edge the result follows.

2. $R^n \quad R \rightarrow R$ transitive

Use the fact that $R^2 \quad R$ and the definition of transitivity. Proof left to the ......

Q. E. D.