Solution to Test #1 (open books/notes)

Part I: Multiple-Choice Problems (60%). Select one correct answer in each problem. Each problem is worth 5 points.

1. Suppose we observe a system for 2000 minutes. Within this period, the busy time of the system is 800 minutes, the total number of customers completed is 10000 and the average number of customers in the system is 3. Which of the following is not true?
   (a) The throughput of the system is 5 customers/minute  
   (b) the wait time per customer is 0.52 minute  
   (c) The service time (not including the waiting time) per customer is 0.08 minute  
   (d) The utilization of the system is 0.45  
   (e) the response time per customer is 0.6 minute

2. Consider a phone booth servicing customers arriving at a rate of 12 customers/hour. Suppose the average duration of a phone conversation is 3 minutes. What is the average wait time (just the queuing time) per customer at the phone booth?
   (a) 4 min  
   (b) 4.5 min  
   (c) 5 min  
   (d) 5.5 min  
   (e) 6 min

3. Consider an M/M/3/8 queueing system with arrival rate of $\lambda$ and service rate of $\mu$ for each server. Let $P_i$ stand for the steady-state probability that the system contains $i$ customers (including both queueing and in service). Which one of the following is false?
   (a) the average number of customers is $\sum_{i=0}^{8} (i \times P_i)$  
   (b) the rejection probability is $P_8$  
   (c) the throughput is $P_1 \times \mu + P_2 \times 2\mu + (P_3 + P_4 + P_5 + P_6 + P_7 + P_8) \times 3\mu$  
   (d) the throughput is $(1 - P_8) \times \lambda$  
   (e) the system utilization is $P_0$
4. Consider a perfect parallel 1-out-of-3 system consisting of three identical components with each component having an independent failure rate of $\lambda$ and an independent repair rate of $\mu$. The Markov model shown above is for the purpose of reliability modeling. Let $P_3(t)$, $P_2(t)$, $P_1(t)$, and $P_0(t)$ be the probabilities that the system is in states 3, 2, 1 and 0, at time $t$, respectively. Which one of the following is false for this Markov model?

(a) $P'_3(t) = -3\lambda P_3(t) + \mu P_2(t)$
(b) $P'_2(t) = 3\lambda P_3(t) - (2\lambda + \mu) P_2(t) + 2\mu P_1(t)$
(c) $P'_0(t) = \lambda P_1(t)$
(d) the reliability of the system at time $t$ is $P_3(t) + P_2(t) + P_1(t)$
√ (e) the availability of the system at time $t$ is $1 - P_0(t)$

5. Continue from the same Markov model. Suppose we assign a reward of one to state 0, and a reward of zero to states 3, 2 and 1. The expected instantaneous reward rate $E[Z(t)]$ under this reward assignment is equal to

(a) the reliability of the system at time $t$
√ (b) the unreliability of the system at time $t$
(c) the mean time to failure of the system
(d) the mean up time in $[0,t]$
(e) none of above

6. Continue from the same Markov model. Suppose we assign a reward of zero to state 0, and a reward of one to states 3, 2 and 1. The expected cumulative reward until absorption $E[Y(\infty)]$ under this reward assignment is equal to

(a) the mean down time in $[0,t]$
(b) the mean up time in $[0,t]$
(c) the mean number of jobs which the system can service during $[0,t]$
√ (d) the mean time to failure
(e) none of above
7. For the same 1 out of 3 system in the last problem, consider a Markov model shown above for the purpose of availability modeling. Let $P_3(t)$, $P_2(t)$, $P_1(t)$, and $P_0(t)$ be the probabilities that the system is in states 3, 2, 1 and 0, at time $t$, respectively. Which one of the following is false for this Markov model?

(a) $P'_3(t) = -3\lambda P_3(t) + \mu P_2(t)$
(b) $P'_2(t) = 3\lambda P_3(t) - (2\lambda + \mu)P_2(t) + 2\mu P_1(t)$
(c) $P'_0(t) = \lambda P_1(t) - 3\mu P_0(t)$
(d) the reliability of the system at time $t$ is $P_3(t) + P_2(t) + P_1(t)$
(e) the availability of the system at time $t$ is $1 - P_0(t)$

8. Continue from the same Markov model. Suppose we assign a reward of zero to state 0, and a reward of one to states 3, 2 and 1. The expected cumulative reward $E[Y(t)]$ under this reward assignment is equal to

(a) the mean down time in $[0,t]$
(b) the mean up time in $[0,t]$
(c) the mean number of jobs which the system can service during $[0,t]$
(d) the mean time to failure
(e) the availability of the system at time $t$

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\begin{array}{c}
3 \xrightarrow{3\lambda} 2 \xleftarrow{\mu} 1 \xrightarrow{\lambda} 0
\end{array}
\]
9. Which of the following statements is true for the fault tree model shown above.
   (a) the system fails when component 1 fails
   (b) the system is alive when component 2 is alive
   (c) the system is alive when components 2, 3 and 4 are alive
   √ (d) the system fails when components 1, 6 and 7 fail
   (e) the system fails when component 1, 3 and 5 fail

10. Suppose that a closed system has two classes of jobs A and B. It is known that the response
time of a class A job is $R_A = 15$, the response time of a class B job is $R_B = 20$, the
throughput for class A jobs is $X_A = 8$ and the throughput for class B jobs is $X_B = 2$. What
is the average response time per job regardless of the job class?
   (a) 15
   √ (b) 16
   (c) 17
   (d) 18
   (e) 19
11. Consider an open QNM shown above that has a product-form solution. As shown in the figure, the external arrival rate to the system is 4 jobs/sec and the service rates at the two centers are 20 jobs/sec and 15 jobs/sec, respectively. When a job departs center 1, it goes to center 2 with probability 1. When a job departs center 2, it completes its service (and thus leaves the system) with probability 0.4 or conversely goes to center 1 with probability 0.6. Which of the following is true for the system described above?
(a) throughput at center 1 is 5 jobs/sec
(b) system throughput is 5 jobs/sec
(c) population at center 2 is 1
(d) response time at center 1 per visit is 0.2 sec
(e) system response time is 0.75 sec

12. Consider the closed QNM below consisting of a terminal center (T) with M terminals and a central subsystem (F, C, D and P). Suppose that the visit counts to centers T, F, C, D, P by a client are $v_T, v_F, v_C, v_D,$ and $v_P,$ respectively, with $v_T = 1.$ The average response times per job per visit at centers T, F, C, D, P are $r_T, r_F, r_C, r_D,$ and $r_P,$ respectively. Which one of the following is not true?
(a) the throughput of the terminal center is $v_T \times M/(v_T r_T + v_F r_F + v_C r_C + v_D r_D + v_P r_P)$
(b) the throughput of center D is $v_D \times M/(v_T r_T + v_F r_F + v_C r_C + v_D r_D + v_P r_P)$
(c) the throughput of the central subsystem is $P_0 \times v_F \times M/(v_T r_T + v_F r_F + v_C r_C + v_D r_D + v_P r_P)$

(d) the average amount of time a terminal job stays in the central subsystem is $v_T r_T + v_F r_F + v_C r_C + v_D r_D + v_P r_P$
(e) the average number of customers in the central subsystem is $(P_0 \times v_F \times M/(v_T r_T + v_F r_F + v_C r_C + v_D r_D + v_P r_P)) \times (v_F r_F + v_C r_C + v_D r_D + v_P r_P)$
1. (15 points.) Consider a system with 8 components connected in a structure as shown in the above block diagram. Assume that component \( i \) has an independent failure rate \( \lambda_i \) and an independent repair rate \( \mu_i \). Identify the minimal cut set for this system. Draw a fault tree model based on the minimal cut set for calculating the availability of the system. Label your fault tree nodes and component names properly. No need to write Sharpe code.

**Ans:**

Minimal cut sets: \( \{1,2\}, \{3,6\}, \{4,5\}, \{1,3,7\}, \{2,6,7\}, \{4,6,8\}, \{3,5,8\}, \{2,5,7,8\}, \{1,4,7,8\} \).

The fault tree is not shown. It has an OR gate at the top with 9 AND gates under it with the 9 minimal cut sets identified above.
2. (10 points.) Refer to the Markov model for HW #2, problem #3 as posted on the class website. Let \( P_{a,b,c} \) be the probability that the system is in state \( a, b, c \) in the steady state, where \( a \) stands for the number of low-priority, low QoS clients, \( b \) stands for the number of low-priority, high QoS clients, and \( c \) stands for the number of high-priority clients. Provide an expression for the high-priority client rejection probability in terms of \( P_{a,b,c} \)'s.

Ans:

High-priority client rejection probability = \( P_{0,0,0} + P_{4,1,0} + P_{4,0,1} + P_{2,1,1} + P_{2,0,2} + P_{0,1,2} + P_{0,0,3} \)
3. (15 points.) Consider a system with 2 CPUs and 3 memory modules that requires at least 1 CPU and 1 memory module to be functioning for the system to be functioning. Assume that the failure rates of a CPU and a memory module are $\lambda_c$ and $\lambda_m$, respectively; the repair rates of a CPU and a memory module are $\mu_c$ and $\mu_m$, respectively. Assume that alive components can still fail regardless of the states of other components. The whole system shares a repair facility which repairs failed components one at a time with the repair priority given to the component type with more failures. When there is a tie, that is, when the number of failed CPUs is the same as the number of failed memory modules, it repairs CPU first.

(a) Draw a Markov model for computing the reliability. Use the representation $(i,j)$ where $i$ is the number of alive memory modules and $j$ is the number of alive CPUs. The initial state is $(3,2)$. Suppose $P_{(i,j)}(t)$ is the probability that the system is in state $(i,j)$ at time $t$. Give an analytical expression for the reliability $R(t)$ at time $t$ in terms of $P_{(i,j)}(t)$’s.

(b) Draw a Markov model for computing the availability. Give an analytical expression for the availability $A(t)$ at time $t$ in terms of $P_{(i,j)}(t)$’s.

Ans: Markov Models are in the next page.

(a) $R(t) = P_{(3,2)}(t) + P_{(2,2)}(t) + P_{(1,2)}(t) + P_{(3,1)}(t) + P_{(2,1)}(t) + P_{(1,1)}(t)$

(b) $A(t) = P_{(3,2)}(t) + P_{(2,2)}(t) + P_{(1,2)}(t) + P_{(3,1)}(t) + P_{(2,1)}(t) + P_{(1,1)}(t)$
(a) Markov Model for Reliability Analysis

(b) Markov Model for Availability Analysis