CS/MATH 3414 SECOND EXAM

Name ______________________________

Show your answers on these pages. Either show your work on these pages or on attached sheets.

1. (15) Let

$$
\begin{aligned}
p(x) &= 12\frac{(x-1)x(x-2)}{6} - 4\frac{(x-3)x(x-2)}{2} + 0\frac{(x-3)(x-1)(x-2)}{-6} - 2\frac{(x-3)(x-1)x}{-2},\\
q(x) &= 12 + 8(x-3) + 4(x-3)(x-1) + 1(x-3)(x-1)x,\\
r(x) &= -2 - 1(x-2) + 3(x-2)x + 1(x-2)x(x-1).
\end{aligned}
$$

Using some simple arithmetic (which must be shown) and without any algebra, explain why p, q, and r are all the same polynomial.

2. (20) Consider the fixed point problem

$$x = \frac{1}{2}\tan\frac{\pi x}{2}$$

on the open interval $0 < x < 1$. Explain both graphically and analytically why fixed point iteration $x_{n+1} = (1/2)\tan(\pi x_n/2)$, $n = 0, 1, 2, \ldots$ does not converge to the unique fixed point $s = 1/2$ in the interval $(0,1)$ for any starting point $x_0 \in (0,1)$, $x_0 \neq 1/2$.

3. (15) Find constants w_0, $\tilde{w}_0$, w_1 such that the integration formula

$$\int_0^1 f(x)x\,dx \approx w_0 f(0) + \tilde{w}_0 f'(0) + w_1 f(1)$$

is exact if $f(x)$ is a polynomial of degree ≤ 2. $w_0 =$ ________, $\tilde{w}_0 =$ ________, $w_1 =$ ________.

4. (10) The first three orthogonal polynomials ψ_0, ψ_1, ψ_2 with respect to the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)(1+x)\,dx$$

are

$$\psi_0(x) = 1, \qquad \psi_1(x) = 9x + \alpha, \qquad \psi_2(x) = 130x^2 - 136x + 25.$$

The constant $\alpha =$ ________________.

5. (10) Suppose that an integration formula $\mathcal{F}(f;h)$ satisfies

$$\int_a^b f(x)\,dx = \mathcal{F}(f;h) + a_1 h + a_2 h^2 + a_3 h^3 + \cdots,$$

where the a_i are constants independent of h. Then an $\mathcal{O}(h^2)$ accurate estimate of $\int_a^b f(x)dx$, in terms of $\mathcal{F}(f;h)$ and $\mathcal{F}(f;h/2)$, is

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6. (10) The best technique for the integral

$$\int_0^1 \left(\sin x^2\right) \ln x \, dx$$

would be ________________.

7. (20) The piecewise linear interpolant $\ell(x)$ to $f(x)$ at $x_1 < x_2 < \cdots < x_n$ is a straight line between x_i and x_{i+1}, $i = 1, \ldots, n-1$, and satisfies $\ell(x_i) = f(x_i)$ for $i = 1, \ldots, n$. Linear B-splines $B_i(x)$ for the breakpoints $x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1}$ are defined for $i = 1, \ldots, n$ by

$$B_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x < x_i, \\ \frac{x - x_{i+1}}{x_i - x_{i+1}}, & x_i \le x < x_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

If we write

$$\ell(x) = \sum_{i=1}^{n} \alpha_i B_i(x)$$

for $x_1 \le x \le x_n$, what are the coefficients α_i? $\alpha_i =$________.