CS/Math 3414 Assignment 10

Solution Sketches

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1. The solution to the initial value problem:

$$x' = x \qquad x(0) = c$$

is given by $x = ce^t$. If the initial value is perturbed by ϵ so that $x(0) = c + \epsilon$, then the solution will be given by $x = (c + \epsilon)e^t$. The error at $t = 10$ would be ϵe^{10} and the error at $t = 20$ would be ϵe^{20}. In other words, the solution would be grossly wrong.

If, on the other hand, the initial vlaue problem was:

$$x' = -x \qquad x(0) = c$$

then the solution is given by $x = ce^{-t}$. In this case, the effect of the perturbation would be negligible.

2. Using Taylor series we write:

$$x(t + h) \quad = \quad x(t) + hx'(t) + \frac{h^2}{2}x''(t) + \frac{h^3}{6}x'''(t)$$

$x(0.1)$ can be calculated as:

$$x(0 + 0.1) \quad = \quad x(0) + 0.1x'(0) + \frac{0.1^2}{2}x''(0) + \frac{0.1^3}{6}x'''(0)$$

$x(0)$ is given in the problem to be 1. So is $x'(0)$ (its value is 2). $x''(0)$ can be calculated from the differential equation: $x'' = x^2e^t + x'$. Thus, $x''(0) = 1^2e^0 + 2 = 3$. Similarly, $x'''(0)$ can be calculated by first finding $x'''(t)$ and then substituting $t = 0$. $x'''(t)$ is just the derivative of the given differential equation: $x''' = 2xx'e^t + x^2e^t + x''$. So, $x'''(0) = 2(2)e^0 + e^0 + 3 = 8$. Putting it all together, we get:

$$\begin{aligned} x(0 + 0.1) &= x(0) + 0.1x'(0) + \frac{0.1^2}{2}x''(0) + \frac{0.1^3}{6}x'''(0) \\ &= 1 + 0.1(2) + [(0.1)^2/2](3) + [(0.1)^3/6](8) \\ &= 1 + 0.2 + 0.015 + 0.00133 \\ &= 1.21633 \end{aligned}$$

3. This question simply involves writing out the Taylor series expansion of $x(t + h)$ and using the initial value at $t = 0$ to determine the value of x in the interval $[0, 1]$. As discussed in class, you can either take 'big steps' (e.g., use $h = 1$ and find $x(1)$ in one shot) or take a lot of 'small steps' (e.g., use $h = 0.01$, first find $x(0.01)$, then use this as the initial value and find $x(0.02)$, and so on). In general, higher order Taylor methods should perform better.

4. In the formulation $x' = f(x, t)$, our f here is $f = (1 + t^2)^{-1}x$. $f_x = (1 + t^2)^{-1} > 0$, so the solution curves diverge from one another as $t \to \infty$.

5. Since the derivative of x is y and vice versa, the Taylor series expansions will involve alternating sequences of x and y. Thus,

$$\begin{array}{rl} x' = y & y' = x \\ x'' = y' = x & y'' = x' = y \\ x''' = x' = y & y''' = y' = x \\ x'''' = y' = x & y'''' = x' = y \\ x''''' = x' = y & y''''' = y' = x \end{array}$$

This gives us the following equation for x:

$$\begin{aligned} x(t+h) &= x + hx' + \frac{1}{2}h^2x'' + \frac{1}{3!}h^3x''' + \frac{1}{4!}h^4x'''' + \frac{1}{5!}h^5x''''' \\ &= x(1 + \frac{1}{2}h^2 + \frac{1}{24}h^4) + y(h + \frac{1}{6}h^3 + \frac{1}{120}h^5) \end{aligned}$$

Similarly,

$$\begin{aligned} y(t+h) &= y + hy' + \frac{1}{2}h^2y'' + \frac{1}{3!}h^3y''' + \frac{1}{4!}h^4y'''' + \frac{1}{5!}h^5y''''' \\ &= y(1 + \frac{1}{2}h^2 + \frac{1}{24}h^4) + x(h + \frac{1}{6}h^3 + \frac{1}{120}h^5) \end{aligned}$$

6. Let x_1 refer to x in the original system of equations and x_2 refer to y in the original system of equations. Let x_3 refer to x', x_4 refer to y' and x_5 refer to y'', again in the original system (why is there no x_6?). This means we can rewrite the given system of equations as:

$$\begin{aligned} x_1' &= x_3 \quad \text{(by definition)} \\ x_2' &= x_4 \quad \text{(by definition)} \\ x_3' &= 3x^2 - 7y^2 + sin(t) + cos(x'y') \quad \text{(from the problem)} \\ &= 3x_1^2 - 7x_2^2 + sin(t) + cos(x_3x_4) \\ x_4' &= y'' \\ &= x_2'' \\ &= x_5 \\ x_5' &= y + x^2 - cos(t) - sin(xy'') \quad \text{(from the problem)} \\ &= x_2 + x_1^2 - cos(t) - sin(x_1x_5) \end{aligned}$$

The net effect of this juggling is that we have managed to convert a system of higher-order differential equations into a system of first-order equations (with more variables). Notice that the left side contains the derivatives and the right side is now free of derivatives. So, the final answer is:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 3x_1^2 - 7x_2^2 + sin(t) + cos(x_3x_4) \\ x_5 \\ x_2 + x_1^2 - cos(t) - sin(x_1x_5) \end{bmatrix}$$

This system can be solved by our usual Taylor series method for a system of first-order equations (see previous answer). But the question only asks us to express the given problem as a system of first-order equations, not solve it.

7. We are given $f = a\,sin(\pi x) + b\,cos(\pi x)$. The least-squares error is given by:

$$E = \sum_{i=0}^{m} (a\,sin(\pi x_i) + b\,cos(\pi x_i) - y_i)^2$$

The goal is to find a and b so that E is minimized.

$$0 = \frac{\partial E}{\partial a} = 2\sum_{i=0}^{m} (a\,sin(\pi x_i) + b\,cos(\pi x_i) - y_i)\,sin(\pi x_i)$$

$$0 = \frac{\partial E}{\partial b} = 2\sum_{i=0}^{m} (a\,sin(\pi x_i) + b\,cos(\pi x_i) - y_i)\,cos(\pi x_i)$$

From the given data, we have:

$$\begin{aligned} 2a &= 2 \\ 3b &= 1 \end{aligned}$$

So, the final solution is $f = sin(\pi x) + (1/3)cos(\pi x)$.

8. The coefficients of the quadratic polynomial $(a + bx + cx^2)$ that best approximates the given data are given by:

$$\begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 18 \end{bmatrix}$$

Solving this system of equations produces: $a = 0$, $b = 2/7$, and $c = 29/35$.

9. The closed form solution of the given differential equation $x' = 4x - 5e^{-t}$ is:

$$x = e^{-t} + ce^{4t}$$

Using the initial condition $x(0) = 1$, we get $c = 0$. So, the true solution is given by $x = e^{-t}$. Here's a MATLAB program to test the sensitivity of the numerical solution of this equation, to perturbations in the initial value. A third order Taylor method is employed.

```
t = [0:0.01:2];
truex = exp(-t);
plot(t,truex,'bo') % true solution
hold on;
axis([0 2 -0.2 2.3]);

% setup the quantities for numerical solution
h = 0.01;
trueinitialvalue = 1;
perturbs = [0.1 0.01 0.001 0.0001 -0.0001 -0.001 -0.01 -0.1];

for p = perturbs
  tvalues(1) = 0;
  computedx = [];
```

```
  computedx(1) = trueinitialvalue + p;
  count = 1;
  for j = 0+h:h:2
        count = count + 1;
        xoft = computedx(count-1);
        xdashoft = 4*xoft - 5*exp(-(j-h));
        xdashdashoft = 4*xdashoft + 5*exp(-(j-h));
        xdashdashdashoft = 4*xdashdashoft - 5*exp(-(j-h));
        computedx(count) = xoft + ...
                         h * xdashoft + ...
                         (h*h/2) * xdashdashoft + ...
                         (h*h*h/6) * xdashdashdashoft;
  end
  plot(t,computedx,'r+');
end
xlabel('t');
ylabel('x');
title('trajectories of solutions of an unstable differential equation');
```

Here's the plot produced. The blue line is the true solution and the red lines are the computed solutions. As can be seen, they diverge as t increases.