CS/Math 3414 Assignment 4
Solution Sketches

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1. This is trivial. Expand $f(2h)$ as a Taylor series around 0. The error term will be given by $-h\,f''(\xi)$ where $\xi \in (0, 2h)$. Thus the formula is $O(h)$.

2. We know that

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \frac{h}{2} f''(x) + \frac{h^2}{6} f'''(x) + \cdots$$

and

$$f'(x) = \frac{f(x) - f(x-h)}{h} - \frac{h}{2} f''(x) + \frac{h^2}{6} f'''(x) - \cdots$$

This gives

$$2f'(x) = \frac{f(x+h) - f(x-h)}{h} + \frac{h^2}{3} f'''(x) + \cdots$$

If we make

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

then the error will be $\frac{h^2}{6} f'''(\xi)$ or $O(h^2)$.

3. In the first expression, $\xi \in (x, x+h)$ while in the second $\xi \in (x-h, x)$. Obviously, these cannot be canceled out.

4. Taking the suggestion from the book, we see that

$$\begin{aligned} D(h) &= f(x+h) - f(x-h) \\ &= 2hf'(x) + \frac{h^3}{3} f'''(x) + \frac{2h^5}{5!} f^{(5)}(x) + \cdots \\ S(h) &= f(x+h) + f(x-h) \\ &= 2f(x) + h^2 f''(x) + \frac{2h^4}{4!} f^{(4)}(x) + \cdots \end{aligned}$$

Now,

$$\begin{aligned} D(2h) &= 4hf'(x) + \frac{8h^3}{3} f'''(x) + \frac{64h^5}{5!} f^{(5)}(x) + \cdots \\ 2D(h) &= 4hf'(x) + \frac{2h^3}{3} f'''(x) + \frac{4h^5}{5!} f^{(5)}(x) + \cdots \end{aligned}$$

This means,

$$D(2h) - 2D(h) = 2h^3 f'''(x) + \frac{60h^5}{120} f^{(5)}(x) + \cdots$$

Or,

$$f'''(x) = \frac{D(2h) - 2D(h)}{2h^3} - \frac{h^2}{4} f^{(5)}(\xi)$$

Similarly, we can express the calculations for $f^{(4)}(x)$ in terms of the S function.

$$\begin{aligned}
S(h) &= 2f(x) + h^2 f''(x) + \frac{h^4}{12} f^{(4)}(x) + \frac{h^6}{360} f^{(6)}(x) + \cdots \\
4S(h) &= 8f(x) + 4h^2 f''(x) + \frac{4h^4}{12} f^{(4)}(x) + \frac{4h^6}{360} f^{(6)}(x) + \cdots \\
S(2h) &= 2f(x) + 4h^2 f''(x) + \frac{16h^4}{12} f^{(4)}(x) + \frac{64h^6}{360} f^{(6)}(x) + \cdots
\end{aligned}$$

Therefore,

$$S(2h) - 4S(h) = -6f(x) + h^4 f^{(4)}(x) + \frac{h^6}{6} f^{(6)}(x) + \cdots$$

Or,

$$S(2h) - 4S(h) + 6f(x) = h^4 f^{(4)}(x) + \frac{h^6}{6} f^{(6)}(x) + \cdots$$

Thus,

$$f^{(4)}(x) = \frac{S(2h) - 4S(h) + 6f(x)}{h^4} - \frac{h^2}{6} f^{(6)}(\xi)$$

Both formulas are thus $O(h^2)$ but the second has a smaller error term.