CS/Math 3414 Assignment 9
Solution Sketches

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1. Let us denote the pieces by:

$$\begin{aligned} S_0(x) &= x^3 & 0 \le x \le 1 \\ S_1(x) &= 0.5(x-1)^3 + a(x-1)^2 + b(x-1) + c & 1 \le x \le 3 \end{aligned}$$

   Since they have to be continuous at 1, this gives $c = 1$. Next, we can impose the continuity of derivatives at 1 to obtain $b = 3$. Finally, the continuity of second derivatives gives us $a = 3$.

2. Let us denote the pieces by $S_0(x)$, $S_1(x)$, and $S_2(x)$. Notice that $S_0(x)$ and $S_2(x)$ are given to be linear but $S_1(x)$ is formulated as a cubic. Since $S_0^{''}(x) = S_2^{''}(x) = 0$, both $S_0^{''}(-1)$ and $S_2^{''}(1)$ are also zero. This implies that $S_1^{''}(-1) = S_1^{''}(1) = 0$. Since $S_1^{''}(x)$ is supposed to be linear, this can happen only if $S_1^{''}(x) = 0$. But this would mean that $S_1(x)$ is linear. Hence the entire spline becomes piecewise linear. In such a case, it can *never* be a natural cubic spline because it will not have continuity in first and second derivatives.

3. The natural cubic spline that interpolates the table is quite simply the straight line (polynomial of degree 1) passing through the points. To see why, notice that the line is given by:

$$p(x) = \frac{(y_1 - y_0)}{(t_1 - t_0)}(x - t_0) + y_0$$

   Now,

   - $p(x)$ is linear in $[t_0, t_1]$ (by construction)
   - $p(x)$ is linear in $[-\infty, t_0]$
   - $p(x)$ is linear in $[t_1, \infty]$
   - the values of $p(x)$ match at the knots (t_0, t_1)
   - $p'(x)$ is continuous (actually constant) throughout $[-\infty, \infty]$
   - $p''(x)$ is continuous (actually zero) throughout $[-\infty, \infty]$

4. Let us denote the cubic interpolating polynomial by

$$p_3(x) = ax^3 + bx^2 + cx + d$$

   Since $p_3(0) = 0$ and $p_3^{'}(0) = 1$ we have $d = 0$ and $c = 1$. Similarly using the other two conditions, we get $a = 0.1927$ and $b = -0.0175$. This gives:

$$p_3(x) = 0.1927x^3 - 0.0175x^2 + x$$

   This is *not* a natural cubic spline since $p''(1) \neq 0$.