Homework 3 Solutions
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1. Represent the "cute word puzzle" as a boolean satisfiability problem.

   For simplicity I choose to use five variables l_1, l_2, l_3, l_4, l_5 each of which is true iff the letter at the given position is a vowel (you can define each of them using a CNF expression if you desire). Hence, there is then one clause which denotes that there must be one letter which is not a vowel: $\neg l_1 \vee \neg l_2 \vee \neg l_3 \vee \neg l_4 \vee \neg l_5$. If we only have this clause, it can get satisfied by making all letters to be consonants! There are also $\frac{5!}{3!2!} = 10$ constraints denoting that there are at least four vowels in the word. These can be stated by observing that, in any two letters, there must be a vowel: $(l_1 \vee l_2) \wedge (l_1 \vee l_3) \wedge (l_1 \vee l_4) \wedge (l_1 \vee l_5) \wedge (l_2 \vee l_3) \wedge (l_2 \vee l_4) \wedge (l_2 \vee l_5) \wedge (l_3 \vee l_4) \wedge (l_3 \vee l_5) \wedge (l_4 \vee l_5)$.

2. "If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned."

   Propositional logic statements of the given facts:

   (1) $Mythical \rightarrow \neg Mortal$
   (2) $\neg Mythical \rightarrow Mortal \wedge Mammal$
   (3) $\neg Mortal \vee Mammal \rightarrow Horned$
   (4) $Horned \rightarrow Magical$

   We show how inference is done by direct application, rather than refutation (which would have been the safer thing to do). We also show a variety of inference rules, for practice (although resolution itself would have been sufficient):

(5)	$\neg Mythical \vee \neg Mortal$	From (1)
(6a)	$(Mythical \vee Mortal)$	From (2)
(6b)	$(Mythical \vee Mammal)$	From (2)
(7)	$\neg Mortal \vee Mammal$	Resolution on (5) and (6b)
(8)	$Horned$	Modus Ponens on (3) and (7)
(9)	$Magical$	Modus Ponens on (4) and (8)

   Thus the unicorn is both magical and horned.

3. Is $P \wedge Q \models P \leftrightarrow Q$ true?

   Intuitively this seems true. On any model for which P and Q are true P will be true iff Q is. Let us use resolution-refutation to prove this. First convert the given statement to clauses: P and Q. Then negate the goal:

   $$\begin{aligned} \neg(P \leftrightarrow Q) &= \neg((P \wedge Q) \vee (\neg P \wedge \neg Q)) \\ &= (\neg P \vee \neg Q) \wedge (P \vee Q) \end{aligned}$$

   Thus we have the following clauses:

   (1) P
   (2) Q
   (3) $\neg P \vee \neg Q$
   (4) $P \vee Q$

   Now we just perform resolution:

(5)	$\neg Q$	Resolution on (1) and (3)
(6)	ϕ	Resolution on (2) and (5)

   Thus our intuition is correct.

4. Problem 7.11

   a. $(\neg x_{1,2} \vee \neg x_{2,1} \vee \neg x_{2,2}) \wedge (x_{1,2} \vee x_{2,1}) \wedge (x_{1,2} \vee x_{2,2}) \wedge (x_{2,1} \vee x_{2,2})$

   b. In order to denote that k of the n neighbors of a particular spot contain mines, create clauses which indicate that there are at least k mines and clauses that indicate that there are not $k+1$ mines around it (obviously one of these two is not needed if $k=0$ or $k=n$). To indicate that there are at least k mines, create $\frac{n!}{k!(n-k)!}$ clauses which have the various combinations of k variables (e.g., if (4,4) has 3 mines surrounding it, one clause would be $x_{2,2} \vee x_{3,2} \vee x_{4,2}$). To indicate that there are not $k+1$ mines around it, create $\frac{n!}{(k+1)!(n-k-1)!}$ similar clauses except now containing the negation of the various variables.

   c. In order to use DPLL to prove that a given space does or does not contain a mine, it suffices to add to the clauses that embody the current knowledge of the board the negation of what we want to prove. Thus to prove that a given space, say (4,4), does not have a mine, simply add a clause saying it does (i.e. $x_{4,4}$) and attempt to satisfy the clauses. If they are not satisfiable then that space does not have a mine in it.

   d. If there are M mines in a board with N spaces, there would need to be $\frac{N!}{M!(N-M)!} + \frac{N!}{(M+1)!(N-M-1)!}$ clauses (again excluding $M=0$ and $M=N$) for the same reasons as in part (b). As this is a prohibitively large number of clauses, we could instead simply modify the DPLL algorithm to obtain the same effect. To do this we can simply keep track of how many variables have been assigned the truth value in the model, and once M variables have been, simply assign every unassigned variable the false value and determine if the model is consistent.

   e. The conclusions for part (c) are not invalidated when the global constraint is taken into account since the additional information does not invalidate any of the original information. The only difference is that now we may be able to make conclusions about spots which before we did not know enough to make conclusive statements about.

   f. A fairly simple example of a long-range dependency would be to consider a $1 \times N$ board which is known to contain only 1 mine. If the first spot we probe reveals that it has one mine surrounding it, then all nonadjacent spaces do not have any mines in them.

5. Show that an arbitrary CNF clause can be written as $(P_1 \wedge ... \wedge P_m) \rightarrow (Q_1 \vee ...Q_n)$

   First note that the order of operations does not matter, so we can write any clause as $(\neg P_1 \vee ... \vee \neg P_m) \vee (Q_1 \vee ... \vee Q_n)$. Now simply convert this to an implication $\neg(\neg P_1 \vee ... \vee \neg P_m) \rightarrow (Q_1 \vee ... \vee Q_n)$, which can be simplified to $(P_1 \wedge ... \wedge P_m) \rightarrow (Q_1 \vee ...Q_n)$.