Homework 6 Solutions
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1. Problem 13.8 The probabilities given:

$$p(\text{TestPositive}|\text{HaveDisease}) = 0.99$$

$$p(\neg\text{TestPositive}|\neg\text{HaveDisease}) = 0.99$$

$$p(\text{HaveDisease}) = 0.0001$$

Now what we want to find is the probability that you have the disease. From Bayes' rule we have

$$p(\text{HaveDisease}|\text{TestPositive}) = \frac{p(\text{TestPositive}|\text{HaveDisease})p(\text{HaveDisease})}{p(\text{TestPositive})}$$

From the information we have

$$\begin{aligned} p(\text{TestPositive}) &= p(\text{HaveDisease})p(\text{TestPositive}|\text{HaveDisease}) \\ &\quad + p(\neg\text{HaveDisease})p(\text{TestPositive}|\neg\text{HaveDisease}) \\ &= 0.0001 \times 0.99 + 0.9999 \times 0.01 \end{aligned}$$

Thus

$$p(\text{HaveDisease}|\text{TestPositive}) = \frac{0.99 \times 0.0001}{0.0001 \times 0.99 + 0.9999 \times 0.01} = 0.0098$$

Which is much lower than you might initially suspect when looking at the probabilities.

2. Problem 13.10
Start with

$$p(A|B, C) = p(A|C)$$

multiply both sides by $p(B|C)$ to get

$$p(A|B, C)p(B|C) = p(A|C)p(B|C)$$

Now using the product rule we have $p(A, B|C) = p(A|B, C)p(B|C)$, thus

$$p(A, B|C) = p(A|C)p(B|C)$$

An isomorphic transformation can be made with the equation $p(B|A, C) = p(B|C)$.

3. Problem 13.13

   a. Which of the following sets are sufficient to calculate $p(h|e_1, e_2)$ with no conditional independence information.

      i. The set $p(E_1, E_2), p(H), p(E_1|H), p(E_2|H)$ is not sufficient. $p(H|E_1, E_2) = \frac{p(E_1,E_2|H)p(H)}{p(E_1,E_2)}$. Thus if we can calculate $p(H|E_1, E_2)$ we can also calculate $p(E_1, E_2|H)$. But we can not calculate this without independence information.

      ii. The set $p(E_1, E_2), p(H), p(E_1, E_2|H)$ is sufficient. $p(H|E_1, E_2) = \frac{p(E_1,E_2|H)p(H)}{p(E_1,E_2)}$.

iii. The set $p(H), p(E_1|H), p(E_2|H)$ is not sufficient. This is a subset of the insufficient set in (i).

b. Which of the following sets are sufficient to calculate $p(h|e_1, e_2)$ if $p(E_1|H, E_2) = p(E_1|H)$.

i. The set $p(E_1, E_2), p(H), p(E_1|H), p(E_2|H)$ is sufficient. $p(H|E_1, E_2) = \frac{p(E_1,E_2|H)p(H)}{p(E_1,E_2)}$. $p(E_1, E_2|H) = p(E_2|H)p(E_1|H, E_2) = p(E_2|H)p(E_1|H)$.

ii. The set $p(E_1, E_2), p(H), p(E_1, E_2|H)$ is sufficient as in part (a).

iii. Here the book uses some somewhat tricky notation without a lot of explanation. The expression $p(A)$ is used to denote the probabilities of all different values of A. Thus if A had a domain of $\{a, b, c\}$ then we would have $p(A = a)$, $p(A = b)$, and $p(A = c)$. As a result the set $p(H), p(E_1|H), p(E_2|H)$ is sufficient. As we have $p(E_1, E_2|H) = p(E_2|H)p(E_1|H)$ we can use normalization over the values of H to calculate $p(E_1, E_2)$. If you do not interpret the notation in that manner then the set is insufficient.

4. Problem 13.14

Write the joint distributions tables as:

X	Y	Z	$p(x, y, z)$
$false$	$false$	$false$	a
$false$	$false$	$true$	b
$false$	$true$	$false$	c
$false$	$true$	$true$	d
$true$	$false$	$false$	e
$true$	$false$	$true$	f
$true$	$true$	$false$	g
$true$	$true$	$true$	h

Now we want to write the equation $p(X = true, Y = true|Z = true) = p(X = true|Z = true)p(Y = true|Z = true)$. This can be converted to the joint distribution table as follows: First, $p(Z = true) = b + d + f + g$. Now we can calculate $p(Y = true|Z = true) = \frac{p(Y=true,Z=true)}{p(Z=true)} = \frac{d+h}{b+d+f+g}$ and $p(X = true|Z = true) = \frac{p(X=true,Z=true)}{p(Z=true)} = \frac{f+h}{b+d+f+g}$. Finally, $p(X = true, Y = true|Z = true) = \frac{p(X=true,Y=true,Z=true)}{p(Z=true)} = \frac{h}{b+d+f+g}$. Thus we can write the final equation

$$\frac{h}{b+d+f+h} = \frac{d+h}{b+d+f+h} \times \frac{f+h}{b+d+f+h}$$

which we can then rewrite as

$$h = \frac{(d+h)(f+h)}{b+d+f+h}$$

and then

$$h(b + d + f + h) = (df + dh + hf + h^2)$$

and finally

$$bh = df$$

Now we want to do the same thing for

$p(X = true, Y = false|Z = true) = p(X = true|Z = true)p(Y = false|Z = true)$,

$p(X = false, Y = true|Z = true) = p(X = false|Z = true)p(Y = true|Z = true)$,
$p(X = false, Y = false|Z = true) = p(X = false|Z = true)p(Y = false|Z = true)$,
$p(X = true, Y = true|Z = false) = p(X = true|Z = false)p(Y = true|Z = false)$,
$p(X = true, Y = false|Z = false) = p(X = true|Z = false)p(Y = false|Z = false)$,
$p(X = false, Y = true|Z = false) = p(X = false|Z = false)p(Y = true|Z = false)$,
and $p(X = false, Y = false|Z = false) = p(X = false|Z = false)p(Y = false|Z = false)$.
We thus get

$$\frac{f}{b+d+f+h} = \frac{(f+h)(b+f)}{(b+d+f+h)^2}$$

or $df = bh$,

$$\frac{d}{b+d+f+h} = \frac{(b+d)(d+h)}{(b+d+f+h)^2}$$

or $df = bh$,

$$\frac{b}{b+d+f+h} = \frac{(b+d)(b+f)}{(b+d+f+h)^2}$$

or $bh = df$,

$$\frac{g}{a+c+e+g} = \frac{(e+g)(c+g)}{(a+c+e+g)^2}$$

or $ag = ce$,

$$\frac{e}{a+c+e+g} = \frac{(e+g)(e+a)}{(a+c+e+g)^2}$$

or $ce = ag$,

$$\frac{c}{a+c+e+g} = \frac{(a+c)(c+g)}{(a+c+e+g)^2}$$

or $ce = ag$, and finally

$$\frac{a}{a+c+e+g} = \frac{(a+c)(a+e)}{(a+c+e+g)^2}$$

or $ag = ce$ for a total of two distinct equations, $ag = ce$ and $bh = df$.

5. The number of possible Bayesian networks on three variables is simply the number of acyclic directed graphs with three nodes. There is one graph with no arrows, six with one arrow, twelve with two arrows, and six with three arrows for a total of 25 possible Bayesian networks. Now to determine the possible distinct distributions we can simply look at the different possible graphs. Again, there is one distribution where no variables are related to each other. There are three where two variables are related to each other since the direction an arrow points is irrelevant. There are six with two arrows: $\frac{p(A,B)p(A,C)}{p(A)}$, $\frac{p(B)p(C)p(A,B,C)}{p(B,C)}$, $\frac{p(A,C)p(B,C)}{p(C)}$, $\frac{p(A)p(B)p(A,B,C)}{p(A,B)}$, $\frac{p(A,B)p(B,C)}{p(B)}$, and $\frac{p(A)p(C)p(A,B,C)}{p(A,C)}$. Finally there is one with three arrows, since if every variable is related to another one we have only one distributions $p(A, B, C)$, for a total of 11 possible distributions.