The Numerical Solutions for an Elliptic Control Problem

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***Abstract*—The aim of our paper is to solve numerically a second order elliptic BVP, using a collocation method. Then, an optimal control problem governed by an elliptic equation is considered and numerical solutions are presented.**

I. Introduction

We first present some theoretical informations about Legendre polynomials and Poisson's equation. Then, we introduce the variational form of Poisson equation and we develop his Ritz-Galerkin approximation. Furthermore, we discuss the connection between a spectral method and a collocation one. Next, numerical solutions are determined. Finally, we solve an optimal control problem, where the state system is a Poisson equation.

A. Legendre Polynomials

Let $\Omega = (-1,1)$. The family of Legendre polynomials $(L_n)_{n\geq 0}$ is defined by:

$$\begin{cases} \text{The degree of } L_n \text{ is n, } (L_i, L_j) = 0,\ i \neq j \\ L_n(1) = 1,\ L_n(-1) = (-1)^n,\ \forall n \in N, \end{cases}$$

where (.,.) represents the inner product in $L^2(\Omega)$.

B. The Galerkin approximation of Poisson's equation

Let us consider the elliptic problem:

$$\begin{cases} -\Delta u &= f, \text{ in } \Omega, \\ u &= 0, \text{ on } \Gamma, \end{cases} \qquad (P)$$

where Γ is the smooth boundary of Ω and $f \in L^2(\Omega)$. It is well-known that this problem has a unique solution.

Let $V = H_0^1(\Omega)$ and consider the bilinear form $a : V \times V \rightarrow \Re$ defined by

$$a(u,v) = \int_\Omega \nabla u \nabla v dx, \quad \forall u, v \in V, \text{ which is symmetric}$$

and positive in the sense that $a(v,v) \geq 0,\ \forall v \in V$. Multiplying the equation from (P) by $v \in V$, we get

$$\begin{cases} \text{Find u} \in \text{V such that} \\ a(u,v) = (f,v),\ \forall v \in V. \end{cases} \qquad (P')$$

The next result is well-known.

Theorem 1: The problem (P') admits a unique solution $u \in H_0^1(\Omega)$ which satisfies the following estimate:

$$||u||_{H^1(\Omega)} \leq c||f||_{L^2(\Omega)}, \text{ for some } c > 0.$$

We now introduce the finite dimensional problem. Let $X_n \subset H_0^1(\Omega)$ be a finite dimensional subspace, with dim$X_n = n$.

$$\begin{cases} \text{Find } u_n \in X_n \text{ such that} \\ a(u_n, v_n) = (f, v_n),\ \forall v_n \in X_n. \end{cases} \qquad (P_n')$$

Let $X_n =$span$\{\varphi_1, \varphi_2, ..., \varphi_n\}$ and $u_n = \sum_{i=1}^n \hat{u}_i \varphi_i$, $v_n = \sum_{j=1}^n \hat{v}_j \varphi_j$. The equation from $(P_n^{'})$ becomes

$$\sum_{i=1}^n \sum_{j=1}^n \hat{u}_i \hat{v}_j a(\varphi_i, \varphi_j) = \sum_{j=1}^n \hat{v}_j (f, \varphi_j).$$

We denote $a_{ij} = a(\varphi_i, \varphi_j),\ i,j = \overline{1,n}, \quad f_i = (f, \varphi_i),\ i = \overline{1,n}$, and we obtain a linear algebraic system $\sum_{j=1}^n a_{ij} \hat{u}_j = f_i,\ i = \overline{1,n}$.

Lemma 1: If u is the solution of (P') and u_n the solution of (P_n'), then there exists a positive constant c, independent of u and X_n, such that $||u - u_n||_{H^1(\Omega)} \leq c||u - v_n||_{H^1(\Omega)}$, $\forall v_n \in X_n$.

II. The collocation method

First, let us introduce the Gauss-Lobatto numerical integration method. Let $-1 = \theta_0 < \theta_1 < ... < \theta_N = 1$, the roots of the polynomial$(1-x^2)L_N^{'}$. Then, for every function $g \in C(\bar{\Omega})$ we have:

$$\int_{-1}^1 g(x)dx = \sum_{j=0}^N \rho_j g(\theta_j) + R_{N+1},$$

where ρ_j are the formula's coefficients and R_{N+1} is the reminder. The coefficients are uniquely determined by

$$\int_{-1}^1 q(x)dx = \sum_{j=0}^N \rho_j q(\theta_j), \forall q \in P_{2N-1}(\Omega), \qquad (1)$$

where P_{2N-1} is the space of all polynomials of degree less or equal to $2N-1$.

Using the Gauss-Lobatto method, we rewrite $(P_n^{'})$ as

$$\left\{ \begin{array}{l} \text{Find } u_N \in P_N^0(\Omega) \text{ such that} \\ a_N(u_N, v_N) = (f, v_N),\ \forall v_N \in P_N^0(\Omega) \end{array} \right. \qquad (\bar{P}_N'),$$

where $a_N(u_N, v_N) = \sum_{j=0}^{N} \rho_j u_N^{'}(\theta_j) v_N^{'}(\theta_j)$, $(f, v_N)_N = \sum_{j=0}^{N} \rho_j f(\theta_j) v_N(\theta_j)$ and $P_N^0(\Omega)$ is the space of all polynomials p of degree $\leq$ N satisfying $p(1) = p(-1) = 0$. Next, using (1) together with the estimation of a_N, we get:

$$a_N(u_N, v_N) = -\sum_{j=0}^{N} \rho_j u_N^{''}(\theta_j) v_N(\theta_j). \qquad (2)$$

In the sequel, we introduce a Lagrange basis for $P_N^0(\Omega)$, so $P_N^0(\Omega) = \text{span}\{\psi_1,\ \psi_2, ..., \psi_{N-1}\}$, with $\psi_i(\theta_j) = \delta_{ij},\ i, j = \overline{1, N-1}$. But we know that $\theta_0 = -1$ and $\theta_N = 1$, so $\psi_i(\theta_0) = \psi_i(\theta_N) = 0,\ i = \overline{1, N-1}$. This is the reason for not counting the functions ψ_0 and ψ_N in the sequel. Then, from equation (2), we obtain the collocation problem

$$\left\{ \begin{array}{l} -u_N^{''}(\theta_i) = f(\theta_i), i = \overline{1, N-1} \\ u_N(-1) = u_N(1) = 0. \end{array} \right. \qquad (PC),$$

Here, the equation $-u^{''} = f$ is satisfied exactly in the collocation points $\theta_i,\ i = \overline{1, N-1}$, which are the roots of the polynomial $L_N^{'}(x)$.

A. Computational issues and numerical results

To solve the algebraic linear system obtained from problem (PC), we need to find out the roots of the polynomial $(1 - x^2)L_N^{'}(x)$. The degree of $L_N^{'}$ is $N - 1$, hence:

$$L_N^{'}(x) = k(x - x_1)(x - x_2)...(x - x_{N-1}).$$

Next, we apply the Newton-Raphson method to get the first root x_1 and we have the following recurrence relation :

$$x_1^{(k)} = x_1^{(k-1)} - \frac{L_N^{'}(x_1^{(k-1)})}{L_N^{''}(x_1^{(k-1)})}.$$

Once determined x_1, we build another polynomial $f(x)$ of order $N - 2$, $f(x) = k(x - x_2)...(x - x_{N-1})$. Applying Newton's technique for $f(x)$, we get x_2 and so on. Thus, after p iterations, we obtain the following polynomial

$$f(x) = \frac{L_N^{'}(x)}{\prod_{i=1}^{p-1}(x - x_i)}$$

and the formula $x_p^{(k)} = x_p^{(k-1)} - \frac{f(x_p^{(k-1)})}{f^{'}(x_p^{(k-1)})}$. Therefore

$$f^{'}(x) = \frac{1}{\prod_{i=1}^{p-1}(x - x_i)}[L_N^{''}(x) - L_N^{'}(x)\sum_{i=1}^{p-1}\frac{1}{(x - x_i)}]$$

and $x_p^{(k)} = x_p^{(k-1)} -$

$$\frac{L_N^{'}(x_p^{(k-1)})}{L_N^{''}(x_p^{(k-1)}) - L_N^{'}(x_p^{(k-1)})\sum_{i=1}^{p-1}\frac{1}{(x_p^{(k-1)} - x_i)}},$$

$p = \overline{1, N-1}$. From the basic recurrence formula for Legendre polynomials (e.g.[3], Chap.1) we get $L_i(x)$, $L_i^{'}(x)$, $L_i^{''}(x)$ and $L_i^{'''}(x)$.

Another issue was how to choose the initial approximations of the roots. For determining the first roots of $L_N^{'}(x)$, we use $x_1^0 = -1$ and for the others $x_i^0 = x_{i-1} + 3 \cdot 10^{-3}$, $i = \overline{2, N-2}$. The stopping conditions were $|x_p^{(k)} - x_p^{(k-1)}| < \varepsilon,\ |L_N^{'}(x_p^{(k)})| < \varepsilon,\ p = \overline{1, N-1}$ where ε represents a prescribed error.

Next, we go to the collocation problem (PC). The solution lies in $P_N^0(-1, 1) = \{p \in P_N(\Omega);\ p(-1) = p(1) = 0\}$=$span\{\psi_1, \psi_2, ..., \psi_{N-1}\}$. We choose $\psi_j(x) = (1 - x^2)L_j^{'}(x),\ j = \overline{1, N-1}$. Therefore $u_N = \sum_{j=1}^{N-1} \hat{u}_j \psi_j$. The equations $-u_N^{''}(\theta_i) = f(\theta_i)$, $\overline{i = 1, N-1}$ lead to the following algebraic linear system:

$$\sum_{j=1}^{N-1}(-\psi_j^{''}(\theta_i))\hat{u}_j = f(\theta_i),\ i = \overline{1, N-1}. \qquad (3)$$

From the definition of ψ_j we get:

$$\psi_j^{'}(x) = -2xL_j^{'}(x) + (1 - x^2)L_j^{''}(x),$$
$$\psi_j^{''} = -2L_j^{'}(x) - 4xL_j^{''}(x) + (1 - x^2)L_j^{'''}(x).$$

The algebraic linear system was solved by Gaussian elimination.

For N=12 and the prescribed error $\varepsilon = 10^{-3}$, we have obtained $x_0 = -0.953$, $x_1 = -0.846$,..., $x_9 = 0.846$, $x_{10} = 0.953$ and $\hat{u}_1 = 1.542$, $\hat{u}_2 = -0.111$, $\hat{u}_3 = 0.007$, $\hat{u}_4 = 0$, $\hat{u}_5 = 0.0008$. The rest of $\hat{u}_j$ are equal with zero. Hence, the solution is :

$$u(x) = 1.542(1 - x^2)L_1^{'}(x) - 0.111(1 - x^2)L_2^{'}(x) + 0.007(1 - x^2)L_3^{'}(x) + 0.008(1 - x^2)L_5^{'}(x).$$

The continuous solution obtained after solving this linear algebraic systems represents an advantage of the collocation method comparing with the finite difference method.

III. A DISTRIBUTED CONTROL PROBLEM

Let us remind that $\Omega = (-1, 1)$, and consider again the problem

$$\left\{ \begin{array}{rl} -\Delta y & = u,\ \text{in } \Omega, \\ y & = 0,\ \text{on } \Gamma, \end{array} \right.$$

We introduce the spaces $V = H_0^1(\Omega)$, $U = H = L^2(\Omega)$, $V^* = H^{-1}(\Omega)$. We recall here, the variational form:

$$(SE) \qquad \left\{ \begin{array}{l} \text{Find } y \in V \text{ such that} \\ a(y, v) = (u, v),\ \forall v \in V. \end{array} \right.$$

The cost functional $J : V \times U \to \Re$ is defined by

$$J(y, u) = \frac{1}{2}||y - y_d||_H^2 + \frac{1}{2}||u||_U^2,$$

where $y_d \in V$ is a given element.

We therefore consider the optimal control problem

(P^*) Minimize $J(y, u)$ for $u \in U$ and $y \in V$ subject to the state equation (SE).

It is well known that problem (P^*) has a unique optimal pair(see [1]).

A. The Optimality Conditions

We introduce the adjoint state $p \in V$ and the adjoint state equation (AE): $a(p,v) = (y - y_d, v)_H,\ \forall v \in V$. Consider also the cost functional $\Phi(u) = J(y,u)$, where y is the solution of (SE) corresponding to u in the right-hand side. Let $[u + \lambda w, y + \lambda\theta]$ be an increment of the optimal pair $[u,y]$. From the optimality condition $\Phi(u+\lambda w) \geq \Phi(u), \forall w \in U$, we obtain after a short calculation

$$(y - y_d, \theta)_H + (u,w)_U + \frac{\lambda}{2}[||w||_U^2 + ||\theta||_H^2] \geq 0.$$

We let λ tend to 0 and we get

$$(\nabla\Phi(u), w) = (y - y_d, \theta)_H + (u,w)_U \geq 0. \qquad (4)$$

We write (SE) for $[u+\lambda w, y+\lambda\theta]$ and for $[u,y]$ and we get by substraction

$$a(\theta, v) = (w,v)_H, \text{ for any } v \in V. \qquad (5)$$

Taking $v = \theta$ in (AE) we have $a(p,\theta) = (y - y_d, \theta)_H$ (6) and taking $v = p$ in (5) we get, using the symmetry of a, $a(p,\theta) = (w,p)_H$. From the equality above and (6) it follows that $(y - y_d, \theta)_H = (p,w)_U$, which is introduced in (4) to obtain $(\nabla\Phi(u), w) = (u+p, w)_U \geq 0$ for any $w \in U$ and therefore $\nabla\Phi(u) = 0$. The optimality conditions for Problem (P^*) are given by (SE), (AE) and $\nabla\Phi(u) = 0$, that is $u+p = 0$. (7)

B. The Galerkin discretization

We introduce the finite dimensional spaces $V_n = P_N \cap V = P_N^0$, $U_n = P_N \cap U$ and the projection operators $\pi_N : V \to V_n$, $\pi_N^U : U \to U_n$. The Galerkin approximation of the state equation (SE) is given by:

$$(SE_N) \quad \begin{cases} \text{Find } y_N \in P_N^0(\Omega) \text{ such that} \\ a(y_N, v_N) = (u_N, v_N),\ \forall v_N \in P_N^0(\Omega). \end{cases}$$

The existence and the uniqueness of the solution y_N may be found in [3]. The cost functional is:

$$J_N(y_N, u_N) = \frac{1}{2}||y_N - \pi_N y_d||_{L^2(\Omega)}^2 + \frac{1}{2}||u_N||_{L^2(\Omega)}^2,$$

and we introduce the optimal control problem.

(P_N^*) Minimize $J_N(y_N, u_N)$ for $u_N \in U_N$ and $y_N \in V_N$ subject to the state equation (SE_N).

Let $p_N \in V_n$ be the adjoint state which satisfies the adjoint state equation (AE_N)

$$a(p_N, v_N) = (y_N - \pi_N y_d, v_N), \quad \text{for any } v_N \in V_N.$$

And we get the optimality condition $u_N + p_N = 0$. (8)
The necessary conditions for optimality for Problem (P_N^*) are given by (SE_N), (AE_N), and (8).

The projection operator π_N is defined for any $v \in H_0^1(\Omega)$ as: $\pi_n v \in P_N^0(\Omega)$, $\int_\Omega (v - \pi_N v)'(x)\varphi_N'(x)dx = 0$, $\forall \varphi_N \in P_N^0(\Omega)$.

TABLE I
THE COST FUNCTIONALS

R	Φ	$\Phi1$
$R = 1$	$\Phi = 500.774$	$\Phi1 = 499.774$
$R = 2$	$\Phi = 487.610$	$\Phi1 = 483.610$
$R = 32$	$\Phi = 1146.26$	$\Phi1 = 122.255$
$R = 64$	$\Phi = 4096.40$	$\Phi1 = 0.39500$
$R = 128$	$\Phi = 16956.9$	$\Phi1 = 572.942$

C. The Numerical algorithm and computational solutions

For the numerical tests, we consider $y_d = (1 - x^2)(x^6 + x^2 + 1)$, which belongs to $P_N^0(\Omega)$. Let us introduce again $P_N^0(-1,1) = span\{\psi_1, \psi_2, ..., \psi_{N-1}\}$, where L_j are the Legendre polynomials. Then, using the collocation method, we know how to determine the solutions for problems (SE_N) and (AE_N).

Next, we present the steps for solving the problem (P_N^*)(we apply the steepest descent algorithm):

Algorithm ALG

(S_0) Choose $u_N^{(0)} \in U_n$; set k=0.

(S_1) Compute $y_N^{(k)}$ from (SE_N).

(S_2) Compute $p_N^{(k)}$ from (AE_N).

(S_3) Compute $g^{(k)} = \nabla\Phi(u_N^{(k)}) = u_N^{(k)} + p_N^{(k)}$, where Φ is the cost functional : $\Phi(u_N) = J_N(y_N, u_N)$.

(S_4) Compute $\rho_k \geq 0$, which is the solution of the minimization problem:

$$min\{\Phi(u_N^{(k)} - \rho_k g^{(k)}),\ \rho_k \geq 0\} =$$
$$min\{\Phi((1-\rho_k)u_N^{(k)} - \rho_k p_N^{(k)}),\ \rho_k \geq 0\}$$

Set $u^{(k+1)} = (1-\rho_k)u_N^{(k)} - \rho_k p_N^{(k)}$.

(S_5) The stopping criterion :

If $|\Phi(u^{(k+1)}) - \Phi(u^{(k)})| < \varepsilon$ then stop

else $k = k+1$; go to (S_1);

It was difficult to find a suitable starting control $u_N^{(0)}$ for the step (S_0). To find a better $u_N^{(0)}$, we choosed a constant control of the form $u_N = R$, with $R \geq 0$. Next, let us denote by $\Phi1$, the cost functional which doesn't contain the term $\frac{1}{2}||u_N||_{L^2(\Omega)}^2$.

In table 1, we see the values of Φ and Φ_1 corresponding to some increasing R and N=21:

We have tried to find better values for Φ. We have used the Azimuth Mark Method(for more details see [4]) with respect to R. We had two strategy for the stopping criterion: one with no interest about the level of u, case in which the ending condition was $|\Phi1(u_N^{(k+1)}) - \Phi1(u_N^{(k)})| < 0.001$(I), and the second one where the quantity of u was taken into account $|\Phi(u_N^{(k+1)}) - \Phi(u_N^{(k)})| < 0.001$(II).

We obtained the following results:

I.-after 25 AMM iterations, $R = 62,332$, $\Phi1_N(u_N^{(0)}) = 0.0256$, $||y_N^{(0)} - y_d||\text{max} = 0.2657$, $\Phi_N(u_N^{(0)}) = 3885.31$.

II.-after 20 AMM iterations, $R = 7.309$, $\Phi_N(u_N^{(0)}) = 509.098$, $||y_N^{(0)} - y_d||\text{max} = 27.2046$, $\Phi1_N(u_N^{(0)}) = 455.669$.

Next, we have modified the search formula for $u_N^{(k+1)}$ due to the low values of $p_N^{(k)}$ and in consequence, the algorithm

ALG changed. Hence, the step (S_3) is no longer counted and (S_4) became:

Compute $\rho_k \geq 0$, which is the solution of the minimization problem: $min\{\Phi(u_N^{(k)} - \rho_k p_N^{(k)}),\ \rho_k \geq 0\}$

Set $u^{(k+1)} = u_N^{(k)} - \rho_k p_N^{(k)}$.

The results obtained using ALG are as follows:

I. In this case, for ALG we also used 2 types of ending conditions

a. $|\Phi 1(u_N^{(k+1)}) - \Phi 1(u_N^{(k)})| < 0.001$ -after 3 iterations:

$$u_N(i) = 62.37,\ 62.47,\ 62.61, ...,\ 62.61,\ 62.47,\ 62.37.$$

$$\Phi 1_N(u_N) = 0.0218,\ ||y_N^{(0)} - y_d||\text{max} = 0.2592,$$

$\Phi_N(u_N) = 3891.85.$

b. $|\Phi(u_N^{(k+1)}) - \Phi(u_N^{(k)})| < 0.001$ -after 2 iterations:

$$u_N(i) = 62.35,\ 62.40,\ 62.46,,\ ...,\ 62.46,\ 62.40,\ 62.35.$$

$$\Phi_N(u_N) = 3884.24,\ ||y_N^{(0)} - y_d||\text{max} = 0.2689$$

$\Phi 1_N(u_N) = 0.02395.$

II.-after 2 iterations:

$$u_N(i) = 7.34,\ 7.43,\ 7.56, ...,\ 7.56,\ 7.43,\ 7.34.$$

$$\Phi_N(u_N) = 453.764,\ ||y_N^{(0)} - y_d||\text{max} = 26.626,$$

$\Phi 1_N(u_N) = 385.931.$

Note the number of iterations for ALG algorithm is very small, thanks to the work done in step S_0 by the AMM. Also note that the structure of the starting control u^0 is changed by ALG.

For a similar algorithm, we refer to [2]. Our next goal is to solve the case when the control is subject to restrictions.

We also solved this optimal control problem in 2D case. This one corresponds to the problem of a membrane fixed at its boundary and loaded by a transversal force $u(x)dx$ by surface unit dx.

IV. Conclusions

The collocation method works for the optimal control problems.The numerical results confirm the theoretical ones, derived from [1] and [3]. Due to the computational results obtained in Section 2 for the roots of the corresponding polynomials, the programming effort is less important than for the Finite Element Method. The numerical results are accurate. We may conclude that the collocation method is an important alternative for FEM.

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