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Numerical Optimal Harvesting for an Age-Dependent Prey-Predator System

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Abstract. In this paper, we develop a numerical study of an optimal harvesting problem for age-dependent prey - predator system. Here, the rates of growth and decay as well as the interaction effect between species are assumed to be depending on age, time and space. Existence, uniqueness and necessary conditions for the optimal control are assured in case of a small final time T . The discrete parabolic nonlinear dynamical systems are obtained by using a finite difference semi-implicit scheme. Then a numerical algorithm is developed to approximate the optimal harvesting effort and the optimal harvest. Results of the numerical tests are given.

Keywords Age structured population dynamics; Numerical Algorithm; Optimal harvesting; Optimality conditions

1 Introduction

This paper is concerned with an optimal harvesting problem for a system consisting of two populations with age dependence, space structure, nonlocal birth process arising as a boundary condition, and interactions of prey - predator type. The use of optimal theory to obtain optimal strategies for the control of biological age-dependent populations was introduced by Rorres and Fair [12], in their early study on a continuous age-structure single species, published more than 25 years ago. Since then, the control theory of age-dependent single species has been intensively researched and well developed. Ainseba and Langlais [1] investigated the controllability of a linear model describing the dynamic of a single species population with age dependence and spatial structure. In [8], Barbu et al. studied the problem of exact controllability of the linear Lotka-McKendrick model of one population dynamics and proved that a closed population is more efficiently controlled through birth control as opposed to migration and eradication. Other optimal control problems for linear systems describing the evolution of an age structured population can be found in [3], [7], [11]. In [4] and [5], the

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7 authors examined optimal harvesting problems of an age dependent population
 8 in a periodic environment. An optimal control problem for a nonlinear age-
 9 dependent population dynamics is described in [2]. The optimal control prob-
 10 lems of age-structured multispecies have been also reported in the literature,
 11 however this topic has come into view just very recently. Fister and Lenhart [9]
 12 investigated the optimal harvesting of an age-dependent competitive population
 13 system, where the existence and unique characterization of the optimal solution
 14 is established by means of Ekeland’s variational principle. Zhao, et al. [13] [14]
 15 treated harvesting problems for two-species system with diffusion.

16 We introduce the following optimal harvesting problem,

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$$(OH) \quad \sup \sum_{i=1}^2 \int_Q [u_i(a, t, x)p_i(a, t, x) - u_i^2(a, t, x)] da dt dx ,$$

for all $u = (u_1(a, t, x), u_2(a, t, x)) \in U$, where the corresponding state variable (p_1, p_2) satisfies the state system,

$$\left\{ \begin{array}{l} \frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} - k_1 \Delta p_1 = -\mu_1(a, x, t)p_1 \\ \quad - \lambda_1(a, x, t)P_2(x, t)p_1 - u_1(a, x, t)p_1, \text{ in } Q, \\ \frac{\partial p_2}{\partial t} + \frac{\partial p_2}{\partial a} - k_2 \Delta p_2 = -\mu_2(a, x, t)p_2 \\ \quad + \lambda_2(a, x, t)p_1(a, x, t)p_2(a, x, t) - u_2(a, x, t)p_2, \text{ in } Q, \\ p_1(0, x, t) = \int_0^A \beta_1(a, x, t, P_2(x, t))p_1(a, x, t) da, \text{ in } Q_T, \\ p_2(0, x, t) = \int_0^A \beta_2(a, x, t, P_1(x, t))p_2(a, x, t) da, \text{ in } Q_T, \\ \frac{\partial p_i}{\partial \nu}(a, x, t) = 0, \text{ on } \Sigma, \\ p_i(a, x, 0) = p_{i0}(a, x), \text{ in } Q_A, \end{array} \right. \quad (1)$$

where $P_i(x, t) = \int_0^A p_i(a, x, t) da$ in Q_T , $i = 1, 2$, $\Omega \in \mathbb{R}^N$ ($N = 1, 2, 3$) is a bounded domain with smooth enough boundary $\partial\Omega$, and ν is the outward unit normal to Ω . The sets introduced in (1) are defined as follows: $Q = (0, A) \times \Omega \times (0, T)$, $Q_T = \Omega \times (0, T)$, $Q_A = (0, A) \times \Omega$, $\Sigma = (0, A) \times \partial\Omega \times (0, T)$. The function p_1 represents the density of the prey population and p_2 the density of the predator population. k_i , $i = 1, 2$ are positive constants and denote the diffusion rates for the two species within Ω . We assume that the two population have the same life expectancy A , $0 < A < +\infty$. μ_i and β_i are the average mortality and fertility rates for the populations p_i , $i = 1, 2$. The functions λ_i , $i = 1, 2$ represent the inter-specific competitions effects. p_{i0} give the initial density distributions

of populations p_i , $i = 1, 2$. u_i are the controls and denote the harvesting rate of populations p_i , $i = 1, 2$.

The set of controls U is given by:

$$U = \{v_1 \in L^2(Q) : 0 \leq v_1(a, x, t) \leq L_1 \text{ a.e. in } Q\} \\ \times \{v_2 \in L^2(Q) : 0 \leq v_2(a, x, t) \leq L_2 \text{ a.e. in } Q\},$$

where $L_i > 0$, $i = 1, 2$, are positive constants.

The cost functional

$$J(u_1, u_2) = \sum_{i=1}^2 \int_Q [u_i(a, t, x)p_i(a, t, x) - u_i^2(a, t, x)] da dt dx \quad (2)$$

represents the profit due to harvesting.

Throughout this paper the following assumptions hold ($i = 1, 2$):

- (A1) $\mu_i \in L_{loc}^\infty([0, A] \times [0, T] \times \bar{\Omega})$, $\mu_i(a, x, t) \geq \mu_0(a, t) \geq 0$, with $\mu_0 \in L_{loc}^\infty([0, A] \times [0, T])$ and $\int_0^A \mu_0(a, t + a - A) da = +\infty$.
- (A2) $\lambda_i \in L^\infty(Q)$ with $0 \leq \lambda_i(a, x, t) \leq M$ a.e. in Q , where M is constant.
- (A3) $\beta_i(\cdot, \cdot, \cdot, s) \in L^\infty(Q)$, for all $s \in \mathbb{R}^+$ and $\beta_i(a, x, t, s)$ is twice continuously differentiable in s , such that

$$0 \leq \beta_i(a, x, t, s) \leq \bar{M}, \left| \frac{\partial \beta_i}{\partial s}(a, x, t, s) \right| + \left| \frac{\partial^2 \beta_i}{\partial s^2}(a, x, t, s) \right| \leq \bar{M}$$

a.e. in $Q \times \mathbb{R}^+$, and \bar{M} is constant.

- (A4) $\beta_1(a, x, t, \cdot)$ is nonincreasing a.e. $(a, x, t) \in Q$ and $\beta_2(a, x, t, \cdot)$ is nondecreasing a.e. $(a, x, t) \in Q$.
- (A5) $p_{i0} \in L^\infty(Q_A)$, $p_{i0}(a, x) \geq 0$ a.e. in Q_A .

The paper is organized as follows. In Section 2, we remind some results about the existence and uniqueness of the system (1) and problem (OH) proved in [14], together with the first order necessary optimality conditions for (OH). Section 3 describes the numerical scheme used for discretizing the state equation (1) and its adjoint system (5). A numerical algorithm to approximate the optimal harvesting effort is derived in Section 4. Section 5 and 6 are dedicated to the numerical experiments and some conclusions, respectively.

2 The optimality conditions

In this section we introduce some theoretical results without proof (e.g. [14]).

Theorem 1. For any $(u_1, u_2) \in U$, system (1) has a unique nonnegative solution $(p_1, p_2) \in L^2(Q) \times L^2(Q) \cap L^\infty(Q) \times L^\infty(Q)$, such that

$$0 \leq p_i(a, x, t) \leq M_1, \text{ a.e. } (a, x, t) \in Q, \quad i = 1, 2,$$

where $M_1 > 0$ is a constant independent of p_i and u_i , $i = 1, 2$.

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7 **Theorem 2.** *If T is small enough, then problem (OH) has a unique optimal*
 8 *control in U .*

10 **Lemma 1.** *Suppose that $u^* = (u_1^*, u_2^*) \in U$ is an optimal control for problem*
 11 *(OH) and (p_1, p_2) denotes the solution of system (1) corresponding to (u_1^*, u_2^*) .*
 12 *Then for any given $v = (v_1, v_2) \in L^\infty(Q) \times L^\infty(Q)$, such that $(u_1^* + \varepsilon v_1, u_2^* +$
 13 *$\varepsilon v_2) \in U$ and for any $\varepsilon > 0$ small enough, the following limits holds:**

$$15 \frac{1}{\varepsilon} \left(p_1^{u^* + \varepsilon v} - p_1, p_2^{u^* + \varepsilon v} - p_2 \right) \rightarrow (z_1, z_2), \quad \text{in } L^2(Q) \times L^2(Q), \quad \text{as } \varepsilon \rightarrow 0^+,$$

17 where $(p_1^{u^* + \varepsilon v}, p_2^{u^* + \varepsilon v})$ is the solution of system (1) corresponding to $(u_1^* +$
 18 $\varepsilon v_1, u_2^* + \varepsilon v_2)$ and (z_1, z_2) satisfies the tangent linear system:

$$19 \left\{ \begin{array}{l} 20 \frac{\partial z_1}{\partial t} + \frac{\partial z_1}{\partial a} - k_1 \Delta z_1 = -(\mu_1 z_1)(a, x, t) - (\lambda_1 p_1)(a, x, t) Z_2(x, t) \\ 21 - \lambda_1 z_1(a, x, t) P_2(x, t) - (u_1 z_1)(a, x, t) - (v_1 p_1)(a, x, t), \text{ in } Q, \\ 22 \frac{\partial z_2}{\partial t} + \frac{\partial z_2}{\partial a} - k_2 \Delta z_2 = -(\mu_2 z_2)(a, x, t) + (\lambda_2 p_2 z_1)(a, x, t) \\ 23 + (\lambda_2 p_1 z_2)(a, x, t) - (u_2 z_2)(a, x, t) - (v_2 p_2)(a, x, t), \text{ in } Q, \\ 24 z_1(0, x, t) = \int_0^A \beta_1(a, x, t, P_2(x, t)) z_1(a, x, t) da \\ 25 + Z_2(x, t) \int_0^A \frac{\partial \beta_1}{\partial s}(a, x, t, P_2(x, t)) p_1(a, x, t) da, \text{ in } Q_T, \\ 26 z_2(0, x, t) = \int_0^A \beta_2(a, x, t, P_1(x, t)) z_2(a, x, t) da \\ 27 + Z_1(x, t) \int_0^A \frac{\partial \beta_2}{\partial s}(a, x, t, P_1(x, t)) p_2(a, x, t) da, \text{ in } Q_T, \\ 28 \frac{\partial z_i}{\partial \nu}(a, x, t) = 0, \text{ on } \Sigma, \\ 29 z_i(a, x, 0) = 0, \text{ in } Q_A, \end{array} \right. \quad (3)$$

30 where $Z_i(x, t) = \int_0^A z_i(a, x, t) da$, $P_i(x, t) = \int_0^A p_i(a, x, t) da$, $i = 1, 2$.

31 **Theorem 3.** *If $u^* = (u_1^*, u_2^*)$ is an optimal control for problem (OH) and*
 32 *(p_1, p_2) is the solution of system (1) corresponding to (u_1^*, u_2^*) , then*

$$33 u_i^* = \min \left\{ \frac{1}{2} (1 + q_i)^+ p_i, L_i \right\}, \quad i = 1, 2,$$

34 where (q_1, q_2) is the solution of the adjoint system (5) corresponding to (u_1^*, u_2^*) .

35 As a consequence of Theorem 3, we give the following result which will be
 36 used for developing the practical algorithm introduced in Section 4:

Remark 1. An optimal control $u^* = (u_1^*, u_2^*)$ of problem (OH) satisfies

$$u_i^* = \begin{cases} L_i, & \text{if } \frac{1}{2}(1 + q_i)p_i \geq L_i, \\ 0, & \text{if } \frac{1}{2}(1 + q_i)p_i \leq 0, \end{cases} \quad i = 1, 2. \quad (4)$$

$$\left\{ \begin{array}{l} \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial a} + k_1 \Delta q_1 = (\mu_1 q_1)(a, x, t) - \beta_1(a, x, t, P_2(x, t))q_1(0, x, t) \\ -q_2(0, x, t) \int_0^A \frac{\partial \beta_2}{\partial s}(a, x, t, P_1(x, t))p_2(a, x, t) da + (\lambda_1 q_1)(a, x, t)P_2(x, t) \\ +u_1(a, x, t)(1 + q_1(a, x, t)) - (\lambda_2 p_2 q_2)(a, x, t), \quad \text{in } Q, \\ \frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial a} + k_2 \Delta q_2 = (\mu_2 q_2)(a, x, t) - \beta_2(a, x, t, P_1(x, t))q_2(0, x, t) \\ -q_1(0, x, t) \int_0^A \frac{\partial \beta_1}{\partial s}(a, x, t, P_2(x, t))p_1(a, x, t) da - (\lambda_2 q_2 p_1)(a, x, t) \\ +u_2(a, x, t)(1 + q_2(a, x, t)) + \int_0^A (\lambda_1 p_1 q_1)(a, x, t) da, \quad \text{in } Q, \\ \frac{\partial q_i}{\partial \nu}(a, x, t) = 0, \quad \text{on } \Sigma, \\ q_i(a, x, T) = 0, \quad \text{in } Q_A, \\ q_i(A, x, t) = 0, \quad \text{in } Q_T, \end{array} \right. \quad (5)$$

where $P_i(x, t) = \int_0^A p_i(a, x, t) da$, $i = 1, 2$.

Formula (4) asserts that u^* is a bang bang control, but it doesn't provide a way to calculate u^* when $\frac{1}{2}(1 + q_i)p_i$ belongs to $(0, L_i)$, $i = 1, 2$. Thus, if $u^* = (u_1^*, u_2^*)$ is the optimal control for problem (OH), then for any given $(v_1, v_2) \in L^\infty(Q) \times L^\infty(Q)$, such that $(u_1^* + \varepsilon v_1, u_2^* + \varepsilon v_2) \in U$, for any $\varepsilon > 0$ small enough, we have:

$$\sum_{i=1}^2 \int_Q \left[(u_i^* + \varepsilon v_i) p_i^{u_i^* + \varepsilon v} - (u_i^* + \varepsilon v_i)^2 - u_i^* p_i + (u_i^*)^2 \right] da dx dt \leq 0,$$

where (p_1, p_2) and $(p_1^{u_i^* + \varepsilon}, p_2^{u_i^* + \varepsilon})$ are the solutions of system (1) corresponding to (u_1^*, u_2^*) and $(u_1^* + \varepsilon v_1, u_2^* + \varepsilon v_2)$, respectively. Next, we get

$$\sum_{i=1}^2 \int_Q \left[u_i^* \frac{p_i^{u_i^* + \varepsilon v} - p_i}{\varepsilon} + v_i \left(p_i^{u_i^* + \varepsilon v} - 2u_i^* - \varepsilon v_i \right) \right] da dx dt \leq 0$$

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7 By Lemma 1, passing to the limit as $\varepsilon \rightarrow 0^+$, we have

$$8 \int_Q \sum_{i=1}^2 \left[u_i^* z_i + v_i (p_i - 2u_i^*) \right] da dx dt \leq 0. \quad (6)$$

9 Multiplying the first two equations from (3) by $z_i (i = 1, 2)$ and integrating over Q , we obtain after some calculation that

$$10 \int_Q \sum_{i=1}^2 u_i^* z_i da dx dt = \int_Q \sum_{i=1}^2 v_i p_i q_i da dx dt \quad (7)$$

11 Equations (6) and (7) imply that

$$12 \int_Q \sum_{i=1}^2 v_i \left[(1 + q_i) p_i - 2u_i^* \right] da dx dt \leq 0, \quad \forall (v_1, v_2) \in L^\infty(Q) \times L^\infty(Q), \quad (8)$$

13 thus, for $\frac{1}{2}(1 + q_i)p_i \in (0, L_i)$, we have $u_i^* = L_i$, for $i = 1, 2$. Combining the last result with Remark 1, we get the following practical formulation for u^* .

14 *Remark 2.* An optimal control $u^* = (u_1^*, u_2^*)$ of problem (OH) satisfies

$$15 u_i^* = \begin{cases} L_i, & \text{if } \frac{1}{2}(1 + q_i)p_i > 0, \\ 0, & \text{if } \frac{1}{2}(1 + q_i)p_i \leq 0, \end{cases} \quad i = 1, 2. \quad (9)$$

16 3 Computational issues for state and adjoint equations

17 The numerical tests presented in this paper were performed in the 1D space case, for $\bar{\Omega} = [0, X]$. The discretization is carried out by finite differences. The grids with equidistant nodes are denoted by:

$$18 0 \leq a_i \leq A, \quad i = \overline{1, M}, \quad 0 \leq x_j \leq X, \quad j = \overline{1, R} \quad 0 \leq t_k \leq T, \quad k = \overline{1, N}.$$

19 Next we denote by Δa , Δx , Δt the age, space, and time step of the network. Let $p_1^k(i, j)$, $p_2^k(i, j)$, $P_1^k(j)$, $P_2^k(j)$ be the approximation of $p_1(a_i, x_j, t_k)$, $p_2(a_i, x_j, t_k)$, $P_1(x_j, t_k)$, $P_2(x_j, t_k)$. In the sequel, by $\lambda_l^k(i, j)$, $\mu_l^k(i, j)$, $u_l^k(i, j)$ we understand $\lambda_l(a_i, x_j, t_k)$, $\mu_l(a_i, x_j, t_k)$, $u_l(a_i, x_j, t_k)$, for $l = 1, 2$. Using the Taylor expansion, we propose a first order in time, second order in space and second order in age semi-implicit scheme. Thus, the discrete representation of the first two equations from the state system (1) at (a_i, x_j, t_k) , for $i = \overline{2, M-1}$, $j =$

$\overline{2, R-1}$, $k = \overline{2, N}$, is:

$$\begin{cases} -\frac{1}{2\Delta a}p_1^k(i-1, j) - \frac{k_1}{(\Delta x)^2}p_1^k(i, j-1) \\ + \left[\frac{1}{\Delta t} + \frac{2k_1}{(\Delta x)^2} + \mu_1^k(i, j) + \lambda_1^k(i, j)P_2^{k-1}(j) + u_1^k(i, j) \right] p_1^k(i, j) \\ - \frac{k_1}{(\Delta x)^2}p_1^k(i, j+1) + \frac{1}{2\Delta a}p_1^k(i+1, j) - \frac{1}{\Delta t}p_1(i, j, k-1) = 0 \\ -\frac{1}{2\Delta a}p_2^k(i-1, j) - \frac{k_2}{(\Delta x)^2}p_2^k(i, j-1) \\ + \left[\frac{1}{\Delta t} + \frac{2k_2}{(\Delta x)^2} + \mu_2^k(i, j) - \lambda_2^k(i, j)p_1^k(i, j) + u_2^k(i, j) \right] p_2^k(i, j) \\ - \frac{k_2}{(\Delta x)^2}p_2^k(i, j+1) + \frac{1}{2\Delta a}p_2^k(i+1, j) - \frac{1}{\Delta t}p_2^{k-1}(i, j) = 0. \end{cases} \quad (10)$$

Of course, for $i = 2$, $i = M-1$, $j = 2$, $j = R-1$, the above equations are changed according with the rate of newborn individuals, life expectancy and boundary conditions.

The nonlinear algebraic system (10), which has $2(M-2)(R-2)$ equations and the same number of unknowns, must be solved for all $k = \overline{2, N}$. We use the Newton-Raphson method for getting the nonlinear system solution. Thus, we introduce the unknowns y_l according to

$$y_l = p_1^k(i, j), \quad i = \overline{2, M-1}, \quad j = \overline{2, R-1}, \quad l = \overline{1, (M-2)(R-2)},$$

$$y_l = p_2^k(i, j), \quad i = \overline{2, M-1}, \quad j = \overline{2, R-1},$$

$$l = \overline{(M-2)(R-2) + 1, 2(M-2)(R-2)}.$$

Next, the system (10) can be rewritten as:

$$f_l(y_1, y_2, \dots, y_{2(M-2)(R-2)}) = 0, \quad l = \overline{1, 2(M-2)(R-2)}, \quad (11)$$

where f_1 represents the first equation of the discretized system (10).

If we denote by $Y = (y_1, y_2, \dots, y_{2(M-2)(R-2)})$, then in the neighborhood of Y , each of the functions f_l can be expanded in Taylor series as

$$f_l(Y + \delta Y) = f_l(Y) + \sum_{d=1}^{2(M-2)(R-2)} \frac{\partial f_l}{\partial y_d} \delta y_d + O(\delta Y^2). \quad (12)$$

By neglecting the terms of order δY^2 and higher, we obtain a set of linear equations for the correction δY , that move each function closer to zero simultaneously:

$$\sum_{d=1}^{2(M-2)(R-2)} a_{ld} \delta y_d = \eta_l, \quad l = \overline{1, 2(M-2)(R-2)}, \quad (13)$$

where

$$a_{ld} = \frac{\partial f_l}{\partial y_d}; \quad \eta_l = -f_l(Y).$$

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This linear system can be solved by a direct method or an iterative one. The corrections are then added to solution vector:

$$y_k^{new} = y_k^{old} + \delta y_k.$$

To start this algorithm, we have to choose an initial estimate of the system's solution (at every time level k), and also a stopping criterion for ending the iteration process, which in our case was defined by $\|\delta Y\|_2 \leq 10^{-3}$. The natural choice for the starting solution at time step k is the solution obtained at $k - 1$.

In the sequel, we present the Jacobian matrix corresponding to the system (11).

$$J = \begin{pmatrix} J^1 & J^2 \\ J^3 & J^4 \end{pmatrix}$$

J^1, J^2, J^3, J^4 are real square matrices of order $(M - 2)(R - 2)$. First we introduce the matrix J^1 :

$$\begin{pmatrix} E_1 & e_2 & 0 & \cdots & \cdots & 0 & e_3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_1 & E_2 & e_2 & 0 & \cdots & \cdots & 0 & e_3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & e_1 & E_3 & e_2 & 0 & \cdots & \cdots & 0 & e_3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & e_1 & E_{R-3} & e_2 & 0 & \cdots & \cdots & 0 & e_3 & 0 & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & e_1 & E_{R-2} & 0 & \cdots & \cdots & \cdots & 0 & e_3 & 0 & \cdots & \cdots \\ e_4 & 0 & \cdots & \cdots & \cdots & 0 & E_{R-1} & e_2 & 0 & \cdots & \cdots & 0 & e_3 & 0 & \cdots \\ 0 & e_4 & 0 & \cdots & \cdots & 0 & e_1 & E_R & e_2 & 0 & \cdots & \cdots & 0 & e_3 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & e_4 & 0 & \cdots & \cdots & 0 & e_1 & E_{l_{11}} & 0 & \cdots & \cdots & \cdots & 0 & e_3 \\ 0 & \cdots & \cdots & 0 & e_4 & 0 & \cdots & \cdots & \cdots & 0 & E_{l_{12}} & e_2 & 0 & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & e_4 & 0 & \cdots & \cdots & 0 & e_1 & E_{l_{13}} & e_2 & 0 & \cdots & \cdots \\ \vdots & \ddots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & e_4 & 0 & \cdots & \cdots & 0 & e_1 & E_{l_{1R-3}} & e_2 & 0 & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & e_4 & 0 & \cdots & \cdots & 0 & e_1 & E_{l_{1R-2}} & e_2 & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & e_4 & 0 & \cdots & \cdots & 0 & e_1 & E_{l_{1R-1}} & \cdots \end{pmatrix},$$

with

$$\begin{cases} e_1 = -\frac{k_1}{dx^2}, e_2 = -\frac{k_1}{dx^2}, J_{l,l+R-2} = e_3 = \frac{1}{2\Delta a}, l = 1, (M - 3)(R - 2). \\ J_{l,l-R+2} = e_4 = -\frac{1}{2\Delta a}, l = R - 1, (M - 2)(R - 2). \\ l_{11} = (M - 3)(R - 2), l_{12} = (M - 3)(R - 2) + 1, l_{1o} = (M - 3)(R - 2) + o - 1, \\ o = 3, R - 1. \\ E_l = \frac{1}{\Delta t} + \frac{2k_1}{(\Delta x)^2} + \mu_1^k(i, j) + \lambda_1^k(i, j)P_2^{k-1}(j) + u_1^k(i, j), \text{ for } \text{mod}(l - 1, R - 2) \neq 0, \\ \text{and } \text{mod}(l, R - 2) \neq 0, l = 1, (M - 2)(R - 2), i = \overline{2, M - 1}, j = \overline{3, R - 2}. \\ E_l = \frac{1}{\Delta t} + \frac{k_1}{(\Delta x)^2} + \mu_1^k(i, j) + \lambda_1^k(i, j)P_2^{k-1}(j) + u_1^k(i, j), \text{ for } \text{mod}(l - 1, R - 2) = 0. \\ \text{or } \text{mod}(l, R - 2) = 0, l = 1, (M - 2)(R - 2) i = \overline{2, M - 1}, j = 2, j = R - 1. \end{cases}$$

By $\text{mod}(l, R)$ we understand the remainder of the division of l by R . Let us denote with qq and rr the quotient and remainder of the division between l and $R - 2$. The next formula gives the relation between l, i, j :

$$\begin{cases} i = qq + 2, j = rr + 1, & \text{if } rr \neq 0, l = \overline{1, (M - 2)(R - 2)}, \\ i = qq + 1, j = R - 1, & \text{if } rr = 0, l = \overline{1, (M - 2)(R - 2)}. \end{cases}$$

Next we describe the configuration of the matrices J^3 and J^4 .

$$J^3 = \begin{pmatrix} F_1 & 0 & \cdots & \cdots & 0 \\ 0 & F_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & F_{(M-2)(R-2)-1} & 0 \\ 0 & \cdots & \cdots & 0 & F_{(M-2)(R-2)} \end{pmatrix},$$

The matrix J^4 has the form:

$$\begin{pmatrix} G_1 & g_2 & 0 & \cdots & \cdots & 0 & g_3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_1 & G_2 & g_2 & 0 & \cdots & \cdots & 0 & g_3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & g_1 & G_3 & g_2 & 0 & \cdots & \cdots & 0 & g_3 & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & g_1 & G_{R-3} & g_2 & 0 & \cdots & \cdots & 0 & g_3 & 0 & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & G_1 & G_{R-2} & 0 & \cdots & \cdots & \cdots & 0 & g_3 & 0 & \cdots \\ g_4 & 0 & \cdots & \cdots & \cdots & 0 & G_{R-1} & g_2 & 0 & \cdots & \cdots & 0 & g_3 & 0 \\ 0 & g_4 & 0 & \cdots & \cdots & 0 & g_1 & G_R & g_2 & 0 & \cdots & \cdots & 0 & g_3 \\ \vdots & \ddots & \ddots & \ddots & 0 & \cdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & g_4 & 0 & \cdots & \cdots & 0 & g_1 & G_{l_{11}} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & g_4 & 0 & \cdots & \cdots & \cdots & 0 & G_{l_{12}} & g_2 & 0 & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & g_4 & 0 & \cdots & \cdots & \cdots & 0 & G_{l_{13}} & g_2 & 0 \\ \vdots & \ddots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & g_4 & 0 & \cdots & \cdots & 0 & g_1 & G_{l_{1R-3}} & g_2 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & g_4 & 0 & \cdots & \cdots & \cdots & 0 & g_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & g_4 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

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with

$$\left\{ \begin{array}{l} F_o = -\lambda_2^k(i, j)y_l, \quad o = 1, (M-2)(R-2), \\ \qquad\qquad\qquad l = (M-2)(R-2) + 1, 2(M-2)(R-2) \\ g_1 = -\frac{k_2}{\Delta x^2}, \quad g_2 = -\frac{k_2}{\Delta x^2}, \quad J_{l, l+R-2} = g_3 = \frac{1}{2\Delta a}, \\ \qquad\qquad\qquad l = (M-2)(R-2) + 1, (2M-5)(R-2). \\ J_{l, l-R+2} = g_4 = -\frac{1}{2\Delta a}, \quad l = (M-1)(R-2) + 1, 2(M-2)(R-2), \\ l_{11} = (2M-5)(R-2), \quad l_{12} = (2M-5)(R-2) + 1, \quad l_{1o} = (2M-5)(R-2) \\ \qquad\qquad\qquad + o - 1, \quad o = 3, R-1. \\ G_l = \frac{1}{\Delta t} + \frac{2k_2}{(\Delta x)^2} + \mu_2^k(i, j) - \lambda_2^k(i, j)y_{l-(M-2)(R-2)} + u_2^k(i, j), \\ \qquad\qquad\qquad \text{for } \text{mod}(l-1, R-2) \neq 0 \text{ and } \text{mod}(l, R-2) \neq 0, \\ l = (M-2)(R-2) + 1, 2(M-2)(R-2), \quad i = \bar{2}, \overline{M-1}, \quad j = \bar{3}, \overline{R-2}. \\ G_l = \frac{1}{\Delta t} + \frac{2k_2}{(\Delta x)^2} + \mu_2^k(i, j) - \lambda_2^k(i, j)y_{l-(M-2)(R-2)} + u_2^k(i, j), \\ \qquad\qquad\qquad \text{for } \text{mod}(l-1, R-2) = 0 \text{ or } \text{mod}(l, R-2) = 0, \\ l = (M-2)(R-2) + 1, 2(M-2)(R-2) \quad i = \bar{2}, \overline{M-1}, \quad j = \bar{2}, j = \overline{R-1}. \end{array} \right.$$

The relation between l, i, j is given by:

$$\left\{ \begin{array}{l} i = qq - M + R - 2, \quad j = rr + 1, \text{ if } rr \neq 0, \\ i = qq - M + R - 3, \quad j = R - 1, \text{ if } rr = 0, \\ l = (M-2)(R-2) + 1, 2(M-2)(R-2). \end{array} \right.$$

To present the matrix J as a hole, we described the variables from J^3 and J^4 using the l index of matrix J . Thus the first line of the matrices J^3 and J^4 corresponds to $l = (M-2)(R-2) + 1$. The structure of J is not fully described until we mention that J^2 is a null matrix.

In the second part of this section we focus on the discrete adjoint system. As we did previously, we introduce $q_1^k(i, j), q_2^k(i, j), \beta_1^k(i, j), \beta_2^k(i, j)$ the approximation of $q_1(a_i, x_j, t_k), q_2(a_i, x_j, t_k), \beta_1(a_i, x_j, t_k, P_2(k, j)), \beta_2(a_i, x_j, t_k, P_1(k, j))$. The Taylor expansion is used to derive the discrete problem corresponding to the adjoint system (5). The discrete version of the first two equations from (5) is a first order in time, second order in space and first order in age scheme due to the age boundary condition at $i = 1$. It approximates the dynamics of the continuous system at (a_i, x_j, t_k) , for $i = \bar{1}, \overline{M-1}, j = \bar{2}, \overline{R-1}, k = \overline{N-1}, 2$. Its form is depicted below:

$$\begin{aligned}
& \left(\begin{aligned}
& -\beta_1^k(i, j)q_1^k(1, j) - \frac{1}{2\Delta a}q_1^k(i-1, j) + \frac{k_1}{(\Delta x)^2}q_1^k(i, j-1) \\
& + \left[-\frac{1}{\Delta t} - \frac{2k_1}{(\Delta x)^2} - \mu_1^k(i, j) - \lambda_1^k(i, j)P_2^k(j) - u_1^k(i, j) \right] q_1^k(i, j) \\
& + \frac{k_1}{(\Delta x)^2}q_1^k(i, j+1) + \frac{1}{2\Delta a}q_1^k(i+1, j) + q_2^k(1, j) \\
& \times \int_0^A \frac{\partial \beta_2}{\partial s}(a, j, k, P_1^k(j))p_2^k(a, j) da + \lambda_2^k(i, j)p_2^k(i, j)q_2^k(i, j) \\
& = -\frac{1}{\Delta t}q_1^{k+1}(i, j) + u_1^k(i, j). \\
& -\beta_2^k(i, j)q_2^k(1, j) - \frac{1}{2\Delta a}q_2^k(i-1, j) + \frac{k_2}{(\Delta x)^2}q_2^k(i, j-1) \\
& + \left[-\frac{1}{\Delta t} - \frac{2k_2}{(\Delta x)^2} - \mu_2^k(i, j) + \lambda_2^k(i, j)p_1^k(i, j) - u_2^k(i, j) \right] q_2^k(i, j) \\
& + \frac{k_2}{(\Delta x)^2}q_2^k(i, j+1) + \frac{1}{2\Delta a}q_2^k(i+1, j) + q_1^k(1, j) \\
& \times \int_0^A \frac{\partial \beta_1}{\partial s}(a, j, k, P_2^k(j))p_1^k(a, j) da = -\frac{1}{\Delta t}q_2^{k+1}(i, j) + u_2^k(i, j) + \\
& \int_0^A (\lambda_1 p_1 q_1)^{k+1}(a, j) da, \quad i = \overline{2, M-1}.
\end{aligned} \right. \tag{14}
\end{aligned}$$

$$\begin{aligned}
& \left(\begin{aligned}
& \left[\beta_1^k(1, j) - \frac{1}{\Delta t} - \frac{1}{\Delta a} - \frac{2k_1}{(\Delta x)^2} - \mu_1^k(1, j) - \lambda_1^k(1, j)P_2^k(j) - u_1^k(1, j) \right] q_1^k(1, j) \\
& + \frac{k_1}{(\Delta x)^2}q_1^k(1, j-1) + \frac{k_1}{(\Delta x)^2}q_1^k(1, j+1) + \frac{1}{\Delta a}q_1^k(2, j) + q_2^k(1, j) \\
& \times \left[\int_0^A \frac{\partial \beta_2}{\partial s}(a, j, k, P_1^k(j))p_2^k(a, j) da + \lambda_2^k(1, j)p_2^k(1, j) \right] \\
& = -\frac{1}{\Delta t}q_1^{k+1}(1, j) + u_1^k(1, j). \\
& \left[\beta_2^k(1, j) - \frac{1}{\Delta t} - \frac{1}{\Delta a} - \frac{2k_2}{(\Delta x)^2} - \mu_2^k(1, j) + \lambda_2^k(1, j)p_1^k(1, j) - u_2^k(1, j) \right] q_2^k(1, j) \\
& + \frac{k_2}{(\Delta x)^2}q_2^k(1, j-1) + \frac{k_2}{(\Delta x)^2}q_2^k(1, j+1) + \frac{1}{\Delta a}q_2^k(2, j) + q_1^k(1, j) \\
& \times \int_0^A \frac{\partial \beta_1}{\partial s}(a, j, k, P_2^k(j))p_1^k(a, j) da = -\frac{1}{\Delta t}q_2^{k+1}(1, j) + u_2^k(1, j) \\
& + \int_0^A (\lambda_1 p_1 q_1)^{k+1}(a, j) da, \quad i = 1.
\end{aligned} \right. \tag{15}
\end{aligned}$$

Combining (14) and (15) with the boundary condition for $j = 2, j = R - 1$ we get the discrete adjoint problem which is a linear algebraic system of $2(M - 1)(R - 2)$ equations and unknowns. Both linear systems (13) and (14-15) were solved using the Gaussian elimination.

4 A Numerical Algorithm

In this section we developed a conceptual algorithm to approximate the solution of the optimal harvesting problem (OH). It is based on the optimality conditions described in Section 2 and a Rosen type algorithm (Algorithm 2.7, Chapter 2, Section 2.5 from [6]).

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Algorithm ALG-R

Step 0 : Choose $u_1^{(0)}, u_2^{(0)}$; set $l := 0$;

Step 1 : Compute $p_1^{(l)}, p_2^{(l)}$ from (10);

Step 2 : Compute $q_1^{(l)}, q_2^{(l)}$ from (14) and (15);

Step 3 : Compute $w_i^{(l)}, i = 1, 2$ given by

$$w_i^l = \begin{cases} L_i, & \text{if } \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)} > 0, \\ 0, & \text{if } \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)} \leq 0, \end{cases} \quad i = 1, 2.$$

Step 4 : Compute $(\phi_l^1, \phi_l^2) \in [0, 1] \times [0, 1]$ which is a solution of the maximization problem

$$\max \left\{ \Phi \left(\phi^1 u_1^{(l)} + (1 - \phi^1) w_1^{(l)}, \phi^2 u_2^{(l)} + (1 - \phi^2) w_2^{(l)} \right); (\phi^1, \phi^2) \in [0, 1] \times [0, 1] \right\},$$

where Φ is the cost functional:

$$\Phi(u_1, u_2) = \sum_{i=1}^2 \int_Q [u_i(a, t, x) p_i(a, t, x) - u_i^2(a, t, x)] da dt dx$$

Set $(u_1^{(l+1)}, u_2^{(l+1)}) := (\phi_l^1 u_1^{(l)} + (1 - \phi_l^1) w_1^{(l)}, \phi_l^2 u_2^{(l)} + (1 - \phi_l^2) w_2^{(l)})$.

Step 5 : The stopping criterion

If $\|u_1^{(l+1)} - u_1^{(l)}\|_\infty + \|u_2^{(l+1)} - u_2^{(l)}\|_\infty < \varepsilon$

then stop

else $l := l + 1$; go to Step 1

Another stopping criterion in Step 5 may be add, namely

$$|\Phi(u_1^{(l+1)}, u_2^{(l+1)}) - \Phi(u_1^{(l)}, u_2^{(l)})| < \varepsilon.$$

Now we make an important remark. If we intend to compute a suboptimal bang-bang control, then the Step 4 of the algorithm must be modified. It is easy to see that a convex combination of two bang-bang functions is not a bang-bang one. In order to keep $u_1^{(l+1)}$ and $u_2^{(l+1)}$ in the class of bang-bang controls, in Step 4, we used a convex combination of switching points (a point where a bang-bang function changes its value) of $u_1^{(l)}, u_2^{(l)}$ and $w_1^{(l)}, w_2^{(l)}$. This idea was introduced by Glashoff and Sachs in [10].

5 Numerical tests

The numerical tests have been performed with the following values of the parameter: $\Omega = (0, 1), A = 1, T = 0.2, k_1 = 0.015, k_2 = 0.01, L_1 = 3, L_2 = 1$.

The average mortality rate and the interaction effects were chosen constant as follows:

$$\mu_1 = 0.1, \mu_2 = 0.08, \lambda_1 = 0.12, \lambda_2 = 0.08, \forall(a, x, t) \in [0, 1] \times [0, 1] \times [0, 0.02].$$

Next, we denote by \bar{Q} the set $[0, 1] \times [0, 1] \times [0, 0.02]$. The fertility functions β_i , $i = 1, 2$ used in our experiments have the following representation:

$$\beta_1(a, x, t) = 1.6 \left[\int_0^A p_2(a, x, t) da \right]^{-1}, \quad \forall(a, x, t) \in \bar{Q},$$

$$\beta_2(a, x, t) = 0.9 \left[\int_0^A p_1(a, x, t) da \right]^{0.9}, \quad \forall(a, x, t) \in \bar{Q}.$$

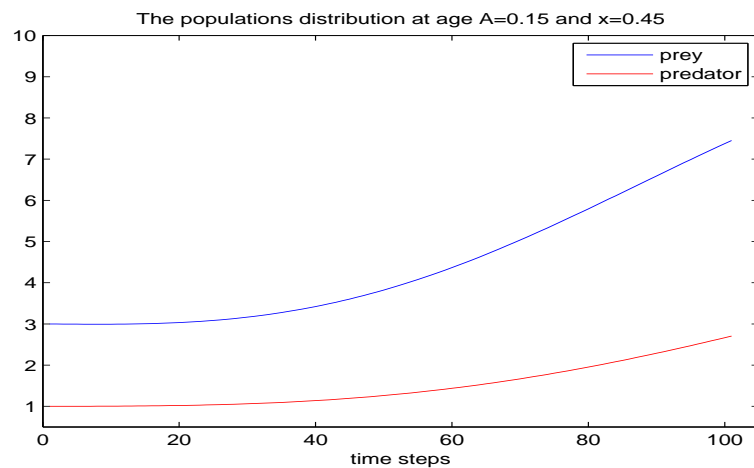


Fig. 1. The evolution in time for populations of age $A = 0.15$, localized at $x = 0.45$.

We considered a spatial discretization of 21 nodes. 21 steps of age have been taken in $[0, A]$. We chose a time step $\Delta t = 0.002$ which corresponds to 101 time steps in the interval $[0, 0.02]$. The stopping criterion parameter ε from the algorithm defined in the previous section was taken 10^{-3} . The initial populations p_{i0} were given by:

$$p_{10}(a, x) = 3, \quad p_{20}(a, x) = 1, \quad \forall(a, x) \in [0, 1] \times [0, 1].$$

The integrals involved in both state and adjoint equations as well as the one from the definition of the cost functional were approximated by Simpson quadrature rule. Figures 1 and 2 illustrate the species distribution in time, for different ages localized at $x = 0.45$, in the absence of any harvest effort. No

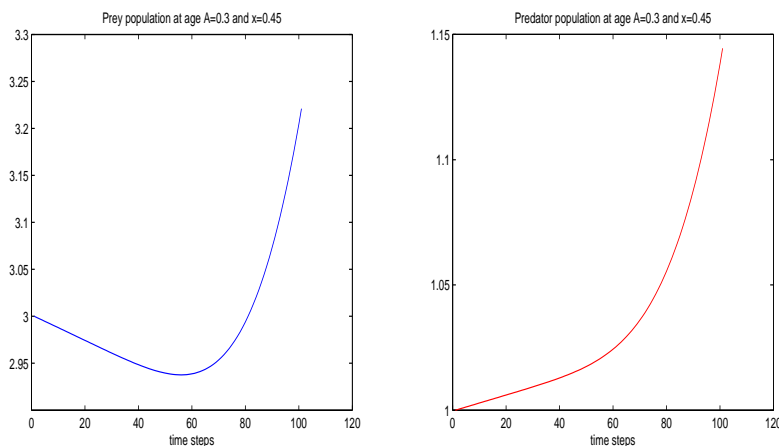


Fig. 2. Time evolution for populations of age $A = 0.3$, at $x = 0.45$.

control influenced the prey and predator space and age distributions at the final time $T = 0.2$, depicted in Figure 3.

Two initial controls have been used to start the algorithm $ALG - R$:

- (1) constant controls of the form $u_i(a, x, t) = 0, i = 1, 2, \forall (a, x, t) \in \bar{Q}$
- (2) constant controls of the form $u_i(a, x, t) = L_i, i = 1, 2, \forall (a, x, t) \in \bar{Q}$

The $ALG - R$ algorithm stopped after 3 and 4 iterations depending on the choice of the initial control, and the corresponding values of the cost functional Φ are given in Table 1.

Table 1. The cost functional values for different starting controls

Iteration	$u_i^{(0)} = 0, \forall (a, x, t) \in \bar{Q}, i = 1, 2$	$u_i^{(0)} = L_i, \forall (a, x, t) \in \bar{Q}, i = 1, 2$
1	0	-0.145167
2	0.192865	0.0692823
3	0.274707	0.120346
4	0.274707	0.275821

Better results were obtained by employing a maximum initial harvesting effort. Thus, it is clear that our maximization algorithm depends on how we choose the starting control. A careful analysis can be done in order to find an optimal initial effort, which may includes alternative controls as initial setting. But here, we focused on other strategy for improving the maximization algorithm. So, we decide to change the Step 3 from algorithm $ALG - R$ according to formula (8). The left term from inequality (8) represents the right directional derivative of

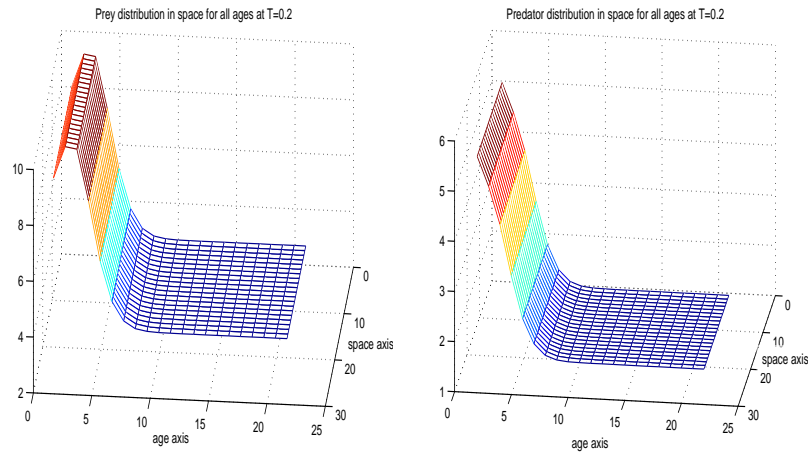


Fig. 3. Populations distribution in space for all ages at $T = 0.2$

the cost functional. By forcing the right directional derivative to get closer to zero, for $\frac{1}{2}(1 + q_i^{(l)})p_i^{(l)} \in (0, L_i)$, $i = 1, 2$, we obtain an updated formula for calculating $w^{(l)}$. Next we introduce this update:

$$w_i^l = \begin{cases} L_i, & \text{if } \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)} \geq L_i, \\ L_i, & \text{if } \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)} \in (0, L_i), \text{ for } L_i - \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)} \leq \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)}, \\ 0, & \text{if } \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)} \in (0, L_i), \text{ for } L_i - \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)} > \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)}, \\ 0, & \text{if } \frac{1}{2}(1 + q_i^{(l)})p_i^{(l)} \leq 0, \quad i = 1, 2. \end{cases}$$

Using this improvement, we started again the maximization algorithm, and after 8 and 9 iterations we obtain the corresponding optimal harvest efforts, for a null and a maximum initial control. The corresponding optimal values are 0.312993 and 0.3186196. The increasing cost functionals are shown in Figure 4.

Figures 5 and 7 describes the space and age distribution for both populations at final time $T = 0.2$, for different starting controls, and Figures 6, 8 show the evolution in time for the prey-predator species at age $A = 0.15$ in $x = 0.45$ and the corresponding optimal harvesting efforts.

6 Conclusions

In this study we focused on the numerical approximation of an optimal harvesting problem for an age-dependent prey-predator model. We derived a numerical algorithm that was implemented in Fortran, under a Linux system, and compiled using ifort compiler. This choice was due to the large dimension of the discrete state vector $y = (p_1, p_2)$, which was $21 \times 21 \times 101 \times 2 = 89082$. Using a semi-implicit scheme, we obtained a nonlinear system of algebraic equations for the state equation and a linear algebraic system for the adjoint equation.

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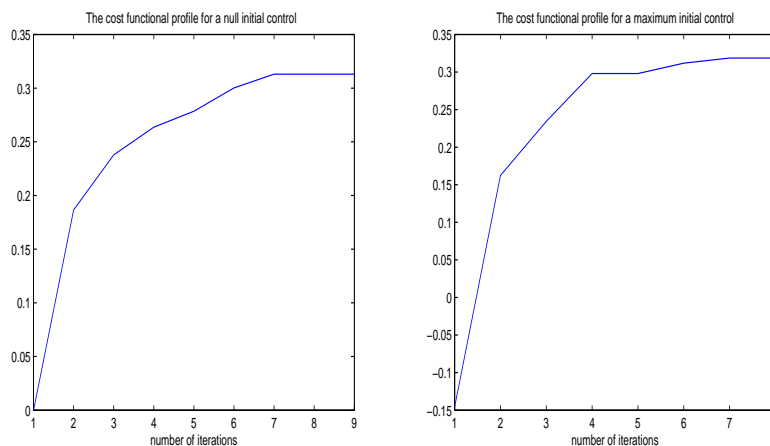


Fig. 4. The cost functional increase evolution for different starting controls using the updated algorithm ALG-R.

The nonlinear system was solved with the Newton-Raphson method, and the solutions to the linear algebraic systems were determined by using the Gaussian elimination.

Several issues including an analysis on choosing better initial controls for the $ALG-R$ as well as the development of a total implicit scheme by approximating the integrals in the state and adjoint systems, with the new terms obtained being included as unknowns in the discrete problems, are under investigation.

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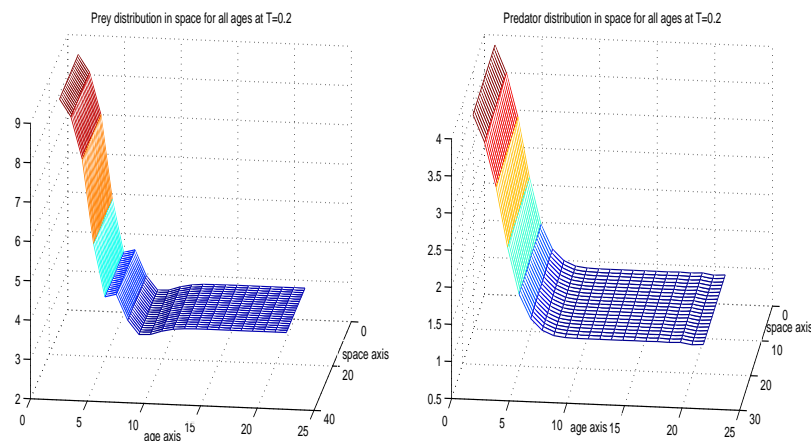


Fig. 5. The populations distribution after applying the optimal harvest effort obtained with a null initial control setting

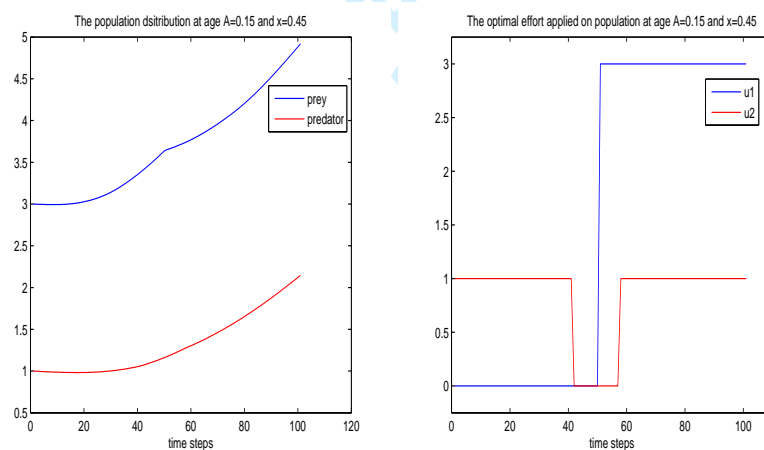


Fig. 6. The populations profile and the corresponding optimal harvest effort obtained with a null starting control setting

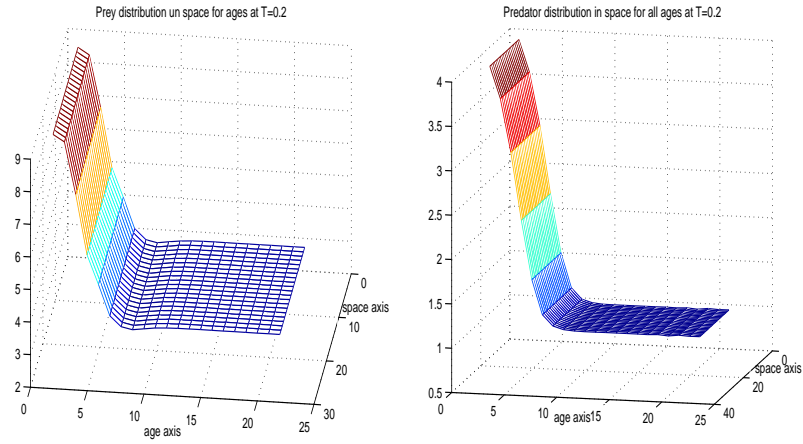


Fig. 7. The populations distribution after applying the optimal harvest effort obtained with a maximum initial control setting

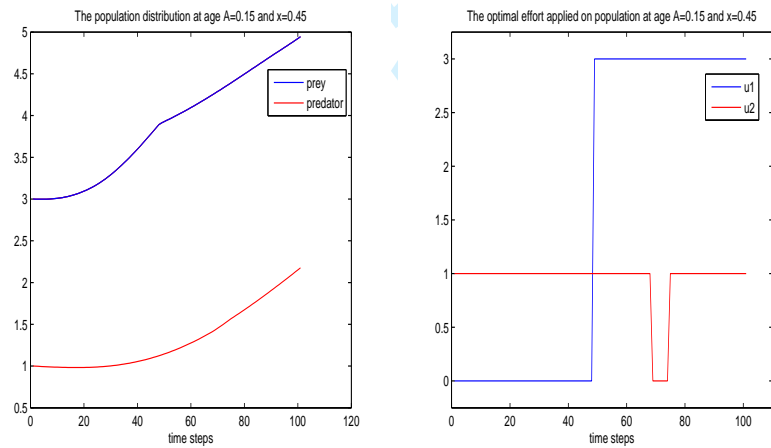


Fig. 8. The populations profile and the corresponding optimal harvest effort obtained with a maximum initial control setting